Examination in Scientific Computing II/NV2

**Time**: 8:00 – 13:00. **Assistance**: Mathematical Handbook (Beta), Calculator.

1. (a) Show that $||Ax||_2 = ||x||_2$ if $A$ is unitary. (2)
   (b) Show that every Hermitian matrix is normal. (2)
   (c) Show that similar matrices have the same eigenvalues but different eigenvectors. (2)
   (d) Of what type is
   - Poisson’s equation $u_{xx} + u_{yy} = f$? (1)
   - the wave equation $u_{tt} = c^2 u_{xx}$? (1)

2. Consider the $2n \times 2n$ matrix

$$
A = \begin{pmatrix}
-n & 0.1 & 0 & 0 & \cdots & 0 \\
0.1 & -n + 1 & 0.1 & 0 & \cdots & 0 \\
0 & 0.1 & n-2 & 0.1 & \cdots & 0 \\
0 & 0 & 0 & 0.1 & n-1 & 0 \\
\end{pmatrix}
$$

(a) Show that the eigenvalues are real and distinct (i.e. have multiplicity at most 1) for all $n \geq 1$. (2)
(b) Propose an iterative algorithm for computing the eigenvector corresponding to the largest eigenvalue. Motivate your choice. (2)
(c) Propose an iterative algorithm for computing the eigenvector corresponding to the smallest eigenvalue. Motivate your choice. (2)
(d) Propose an iterative algorithm for computing the eigenvector corresponding to the eigenvalue closest to the origin. Motivate your choice. (2)

3. Let 

$$L = C[0,2],$$

i.e. the set of continuous functions on the interval $[0,2]$, and 

$$M = \text{span}\{1, x\},$$

i.e. the set of linear functions. Define the scalar product 

$$(f, g) = \sum_{i=0}^{2} f(i)g(i).$$

(a) Determine an ON-basis in $M$. (2)
(b) Compute the orthogonal projection $g^*$ of $f(x) = x^3$ on $M$. (2)
(c) Show that the error $f - g^*$ is orthogonal to $M$. (2)
(d) Show that $F(g) = ||f - g||$ is minimised when $g = g^*$. (2)
4. Consider the differential equation
\[ u_t + u_x = 0, \quad 0 < x < 1, \quad t > 0, \]
\[ u(0, t) = u(1, t), \]
\[ u(x, 0) = f(x), \]
and the finite difference approximation
\[ \frac{v^{n+1}_j - v^n_j}{k} + D_j v^n_j = 0, \quad j = 1, \ldots, N, \quad n = 0, 1, \ldots \]
\[ v^n_0 = v^n_N, \]
\[ v^0_j = f(x_j). \]
You can assume both problems to have unique solutions. Let \( ||u|| = \sqrt{(u, u)} \), where
\[ (f, g) = \int_0^1 f(x)g(x)dx, \]
and \( ||v||_h = \sqrt{(v, v)_h} \), where
\[ (v, w)_h = \sum_{j=1}^N v_j w_j h. \]

(a) Show that the solution to the differential equation satisfies \( ||u|| = ||f|| \) for all \( f \) and \( t > 0 \). (2)
(b) Show that the difference approximation is consistent. (2)
(c) Derive a stability condition. (2)
(d) Give a condition for convergence. Motivate your answer. (2)

5. Show that the solution to
\[ \frac{v_{i+1,j} - 2v_{i,j} + v_{i-1,j}}{h^2} + \frac{v_{i,j+1} - 2v_{i,j} + v_{i,j-1}}{h^2} = 0, \quad (i, j) \in \Omega, \]
\[ v_{i,j} = g(x_i, y_j), \quad (i, j) \in \partial\Omega \]
assumes its maximum on the boundary \( \partial\Omega \). (8)