All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

1. a) Show that for every $x$ and $y$ such that $||x||_2=||y||_2$, there exists a Householder-matrix $P$ such that $Px = y$. (3p.)

b) Show how Householder-transformations can be used to $QR$-factorize a matrix $A$. (3p.)

c) Consider an overdetermined system of equations

$$Ax = b,$$

where $A$ is a real $M \times N$-matrix, $M > N$. One way to solve (1) is to $QR$-factorize $A$ and use that

$$||Ax - b||_2 = ||Rx - Q^Hb||_2.$$  (2)

Show that (2) is true. (2p.)

2. We want to solve the following PDE

$$u_t = u_x, \quad x \in (0,1), \quad t \geq 0,$$

$$u(x,0) = \cos(2\pi x),$$

$$u(0,t) = u(1,t).$$

a) Consider

$$u_j^{n+1} = u_j^n + \lambda (u_{j+1}^n - u_j^n) + \frac{\lambda^2}{2}(u_{j+1}^n - 2u_j^n + u_{j-1}^n)$$

as an approximation to the PDE. Here $\lambda = k/h$, where $k$ and $h$ are the steps in time and space respectively. Derive the truncation error. (4p.)

b) Also consider the approximation

$$v_j^{n+1} = u_j^n + \lambda(u_{j+1}^n - u_j^n)$$

with periodic boundary conditions. Derive necessary and sufficient conditions on $\lambda$ for stability. (4p.)
3. We want to solve

\[ Ax = b \]  

where \( A \) is an \( n \times n \)-matrix defined by

\[
A = \begin{pmatrix}
1 & 1/3 \\
1/3 & 1 & 1/3 \\
& \ddots & \ddots & \ddots \\
& & 1/3 & 1 & 1/3 \\
& & & 1/3 & 1
\end{pmatrix}
\]

and

\[ b = (1, 1, \ldots, 1)^T. \]

An iterative method to solve (3) can be written as

\[ x^{k+1} = Mx^k + c \]  \hspace{1cm} (4)

a) Define and compute \( M \) and \( c \) in (4) when Jacobi’s method is used to solve (3). (4p.)

b) Show that Jacobi’s method on (3) will converge. (4p.)

4. Consider the following boundary-value ODE

\[ u''(x) = f(x), \quad 0 \leq x \leq 1, \]
\[ u(0) = 0, \quad u(1) = 0. \]  \hspace{1cm} (5)

We will use the finite element method to solve (5) on the grid \( x_i = i \cdot h, \)
\( i = 1, \ldots, N, \) \( h = 1/(N + 1). \)

a) Derive the variational formulation to (5). (3p.)

b) Use the piecewise linear hat-functions

\[
\phi_j(x_i) = 1, \quad i = j \\
\phi_j(x_i) = 0, \quad i \neq j, \quad j = 1, \ldots, N.
\]

and the approximation

\[ u = \sum_{j=1}^{N} c_j \phi_j(x) \]

to obtain the resulting linear system of equations \( A\bar{c} = b, \) where \( \bar{c} = (c_1, \ldots, c_N)^T. \) Compute the coefficient-matrix \( A. \) You shall also compute \( b \) for the particular case \( f(x) \equiv 1. \) (5p.)
5. A circulant matrix $C$ is a dense $N \times N$-matrix defined by

$$C = \begin{pmatrix}
c_0 & c_1 & \cdots & c_N \\
c_N & c_0 & c_1 & \cdots \\
\vdots & \vdots & \ddots & \vdots \\
c_1 & \cdots & \cdots & c_0
\end{pmatrix}, \quad (6)$$

i.e. the elements “circulate” along the diagonals. It can be shown that the matrix defined in (6) has an eigenvalue-decomposition as $C = F \Lambda F^{-1}$, where

$$F_{j,k} = \frac{1}{\sqrt{N}} e^{\frac{2\pi i (j-1)(k-1)}{N}}$$

and $\Lambda$ is a diagonal matrix with the eigenvalues of $C$ on the diagonal. The Fourier-matrix $F$ is unitary.

a) Show that the system of equations

$$Cx = b \quad (7)$$

can be solved by the following steps:

(a) $z = F^H b$,
(b) $y = \Lambda^{-1} z$,
(c) $x = F y$. \hfill (2p.)

b) Show that a straight-forward implementation of the algorithm defined in a) requires $O(N^2)$ arithmetic operations. \hfill (3p.)

c) Suggest a better way to solve (7) that requires only $O(N \log_2 N)$ arithmetic operations. Describe (without details) this algorithm. \hfill (3p.)

Good luck! Lina, Lars, Henrik, and Per.