Time: 15:00 – 20:00  
Tools: Pocket calculator, Beta Mathematics Handbook  
Maximum number of points: 40  

All your answers must be very well argued. Calculations shall be demonstrated in detail. Solutions that are not complete can give points if they include some correct thoughts.

1. (a) Let $a = (1\ 1\ 2)^T$ and $b = (3\ 0\ 1)^T$. Compute $ba^Tb$. (2p.)  
(b) Show that every Householder matrix is symmetric. (2p.)  
(c) Compute the spectral radius of  
\[
\begin{pmatrix}
3 & -1 \\
3 & 0
\end{pmatrix}
\]  
(d) Show that a system of equations has a unique solution if the coefficient matrix has full rank. (2p.)

2. (a) Compute the best possible approximation of the function $f(x) = 1$ on the form $g(x) = ax + bx^2$ in the sense of least squares with respect to the scalar product 
\[
(f,g) = \int_{-1}^{1} f(x)g(x) \, dx.
\]
Illustrate your solution with a graph. (4p.)  
(b) Solve the overdetermined system of equations  
\[
\begin{aligned}
x_1 - x_2 &= 0 \\
x_1 + x_2 &= 0 \\
x_2 &= 1
\end{aligned}
\]  
Illustrate your solution with a graph. (4p.)

3. We are interested in the eigenvalues of the following matrix  
\[
A = \begin{pmatrix}
7 & 2 & 0 \\
2 & 15 & 1 \\
0 & 1 & -1
\end{pmatrix}
\]
a) Estimate the eigenvalues of $A$.  

b) Compute one step with the inverse power method and the initial guess $x_0 = (0,0,1)^T$ to determine the smallest eigenvalue of $A$. Which is the eigenvalue and eigenvector approximation? 

c) Compute one step with the inverse power method with a suitable shift and the initial guess $x_0 = (1,0,0)^T$ to determine the intermediate eigenvalue of $A$. Which is the eigenvalue and eigenvector approximation? 

d) Show that the inverse power method converges to the eigenvalue with the smallest absolute value. You can use known convergence results for the power method itself. 

4. Consider the following linear system of equations

$$Au = b,$$  

where

$$A = \begin{pmatrix} 2 & -0.5 & 0 \\ -0.5 & 2 & -0.5 \\ 0 & -0.5 & 2 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. $$

(a) Compute $M$ and $c$ in $u^{k+1} = Mu^k + c$ using Jacobi’s method to solve (1) with (2). 

(b) Show that Jacobi’s method on (1) with (2) will converge. 

(c) Perform one step with Jacobi’s method on (1) with (2) using initial guess $u^0 = (1 \ 1 \ 1)^T$. 

5. We are interested in solving the following partial differential equation

$$u_t = u_x, \quad t > 0, \quad 0 \leq x \leq 1,$$  

using finite differences. So called adaptive methods aim at placing gridpoints where they are “best needed”, i.e. where the discretization error is largest if we use an equidistant cartesian grid

$$x_i = ih, \quad i = 1, \ldots, N,$$

$$t_k = k\Delta t, \quad k = 1, \ldots,$$

where $h$ and $\Delta t$ denote the space- and time-step respectively. 

(a) Consider a nonequidistant grid in space that could be the result from using an adaptive method, see Figure 1.

Figure 1: The computational grid around $x_i$. 

Let $v^n_i$ denote an approximation of $u(x_i, t_n)$. Derive the coefficients $a$, $b$ and $c$ in

$$Qv^n_i = av^n_{i-1} + bv^n_i + cv^n_{i+1}, \quad i = 1, \ldots, N$$

such that $Qv^n_i$ is a second-order approximation of $u_x(x_i, t_n)$. (4p.)

(b) Next we consider adaptivity in time. In order to decide how to choose the time-steps, we want to estimate $\tau_i^n$, the local discretization error in time. We will estimate $\tau_i^n$ by using two different time-stepping schemes that computes the approximations $v^n_{i,I}$ and $v^n_{i,E}$. We know that the computed solutions $v^n_{i,I}$ and $v^n_{i,E}$ fulfill

$$u(x_i, t_n) - v^n_{i,I} \approx c_I \Delta t^3 u_{III}(x_i, t_n),$$

$$u(x_i, t_n) - v^n_{i,E} \approx c_E \Delta t^3 u_{III}(x_i, t_n),$$

where $c_I$ and $c_E$ are known constants. We also know that the following holds for $\tau_i^n$

$$\tau_i^n \approx -\alpha c_I \Delta t^2 u_{III}(x_i, t_n),$$

where $\alpha$ is a known constant. Derive an expression for how $\tau_i^n$ can be estimated from $v^n_{i,I}$ and $v^n_{i,E}$. (4p.)

Good luck! Lina, Per, and Henrik.