# Introduction to Bottom-Up Parsing

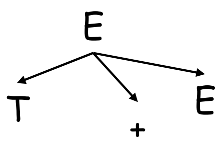
## Outline

- Review LL parsing
- Shift-reduce parsing
- The LR parsing algorithm
- Constructing LR parsing tables



## Top-Down Parsing: Review

- Top-down parsing expands a parse tree from the start symbol to the leaves
  - Always expand the leftmost non-terminal




int * int + int

$E \rightarrow T + E \mid T$
$T \rightarrow (E) \mid int \mid int * T$



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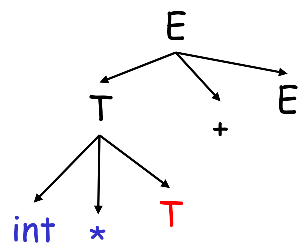


- The leaves at any point form a string $\beta A \gamma$
  - $\beta$ contains only terminals
  - The input string is $\beta b \delta$
  - The prefix $\beta$ matches
  - The next token is b

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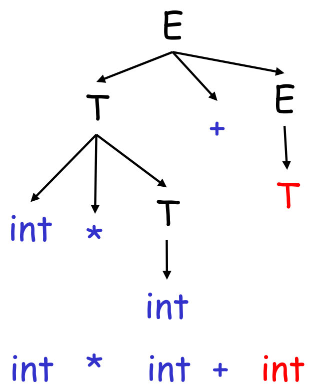


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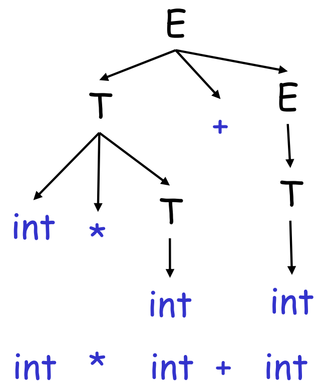


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  - The input string is $\beta b\delta$
  - The prefix $\beta$ matches
  - The next token is b



## Predictive Parsing: Review

- A predictive parser is described by a table
  - For each non-terminal $A$ and for each token b we specify a production $A \to \alpha$
  - When trying to expand $A$ we use $A \to \alpha$ if b follows next
- Once we have the table
  - The parsing algorithm is simple and fast
  - No backtracking is necessary



## Constructing Predictive Parsing Tables

Consider the state $S \to^* \beta A\gamma$

- With b the next token
- Trying to match $\beta b\delta$

There are two possibilities:

1. Token b belongs to an expansion of $A$
   - Any $A \to \alpha$ can be used if b can start a string derived from $\alpha$
   - We say that $b \in First(\alpha)$

Or…

## Constructing Predictive Parsing Tables (Cont.)

2. Token b does not belong to an expansion of $A$
   - The expansion of $A$ is empty and b belongs to an expansion of $\gamma$
   - Means that b can appear after $A$ in a derivation of the form $S \to^* \beta A b \omega$
   - We say that $b \in \text{Follow}(A)$ in this case

   - What productions can we use in this case?
     - Any $A \to \alpha$ can be used if $\alpha$ can expand to $\varepsilon$
     - We say that $\varepsilon \in \text{First}(A)$ in this case



## Computing First Sets

Definition

$$\text{First}(X) = \{ b \mid X \to^* b\alpha \} \cup \{ \varepsilon \mid X \to^* \varepsilon \}$$

Algorithm sketch

1. $\text{First}(b) = \{ b \}$
2. $\varepsilon \in \text{First}(X)$ if $X \to \varepsilon$ is a production
3. $\varepsilon \in \text{First}(X)$ if $X \to A_1 \ldots A_n$ and $\varepsilon \in \text{First}(A_i)$ for $1 \le i \le n$
4. $\text{First}(\alpha) \subseteq \text{First}(X)$ if $X \to A_1 \ldots A_n\ \alpha$ and $\varepsilon \in \text{First}(A_i)$ for $1 \le i \le n$



## First Sets: Example

- Recall the grammar

$$E \to T\,X \qquad X \to +\,E \mid \varepsilon$$
$$T \to (\,E\,) \mid \text{int}\ Y \qquad Y \to *\,T \mid \varepsilon$$

- First sets

$\text{First}(\ (\ ) = \{\ (\ \}$

$\text{First}(\ )\ ) = \{\ )\ \}$

$\text{First}(\text{ int}) = \{ \text{ int } \}$

$\text{First}(\ +\ ) = \{\ +\ \}$

$\text{First}(\ *\ ) = \{\ *\ \}$

$\text{First}(\ T\ ) = \{ \text{ int}, (\ \}$

$\text{First}(\ E\ ) = \{ \text{ int}, (\ \}$

$\text{First}(\ X\ ) = \{\ +, \varepsilon\ \}$

$\text{First}(\ Y\ ) = \{\ *, \varepsilon\ \}$



## Computing Follow Sets

- Definition

$$\text{Follow}(X) = \{ b \mid S \to^* \beta\, X\, b\, \delta \}$$

- Intuition
  - If $X \to A\ B$ then $\text{First}(B) \subseteq \text{Follow}(A)$ and $\text{Follow}(X) \subseteq \text{Follow}(B)$
  - Also if $B \to^* \varepsilon$ then $\text{Follow}(X) \subseteq \text{Follow}(A)$
  - If $S$ is the start symbol then $\$ \in \text{Follow}(S)$

## Computing Follow Sets (Cont.)

<u>Algorithm sketch</u>

1. $\$ \in$ Follow(S)
2. First($\beta$) - $\{\varepsilon\} \subseteq$ Follow(X)
   - For each production $A \to \alpha\ X\ \beta$
3. Follow(A) $\subseteq$ Follow(X)
   - For each production $A \to \alpha\ X\ \beta$ where $\varepsilon \in$ First($\beta$)



## Follow Sets: Example

- Recall the grammar

$E \to T\ X \qquad X \to +\ E \mid \varepsilon$

$T \to (\ E\ ) \mid \text{int}\ Y \qquad Y \to *\ T \mid \varepsilon$

- Follow sets

Follow( + ) = { int, ( }    Follow( * ) = { int, ( }

Follow( ( ) = { int, ( }    Follow( E ) = { ), $ }

Follow( X ) = { $, ) }    Follow( T ) = { +, ) , $ }

Follow( ) ) = { +, ) , $ }    Follow( Y ) = { +, ) , $ }

Follow( int ) = { *, +, ) , $ }



## Constructing LL(1) Parsing Tables

- Construct a parsing table T for CFG G

- For each production $A \to \alpha$ in G do:
  - For each terminal $b \in$ First($\alpha$) do
    - T[A, b] = $\alpha$
  - If $\varepsilon \in$ First($\alpha$), for each $b \in$ Follow(A) do
    - T[A, b] = $\alpha$
  - If $\varepsilon \in$ First($\alpha$) and $\$ \in$ Follow(A) do
    - T[A, $] = $\alpha$



## Constructing LL(1) Tables: Example

- Recall the grammar

$E \to T\ X \qquad X \to +\ E \mid \varepsilon$

$T \to (\ E\ ) \mid \text{int}\ Y \qquad Y \to *\ T \mid \varepsilon$

- Where in the line of Y we put $Y \to * T$ ?
  - In the lines of First(*T) = { * }

- Where in the line of Y we put $Y \to \varepsilon$ ?
  - In the lines of Follow(Y) = { $, +, ) }

## Notes on LL(1) Parsing Tables

- If any entry is multiply defined then G is not LL(1)
  - If G is ambiguous
  - If G is left recursive
  - If G is not left-factored
  - And in other cases as well

- For some grammars there is a simple parsing strategy: *Predictive parsing*
- Most programming language grammars are not LL(1)
- Thus, we need more powerful parsing strategies



## Bottom Up Parsing

## Bottom-Up Parsing

- Bottom-up parsing is more general than top-down parsing
  - And just as efficient
  - Builds on ideas in top-down parsing
  - Preferred method in practice

- Also called LR parsing
  - L means that tokens are read left to right
  - R means that it constructs a rightmost derivation !



## An Introductory Example

- LR parsers don't need left-factored grammars and can also handle left-recursive grammars

- Consider the following grammar:

$$E \rightarrow E + ( E ) \mid int$$

  - Why is this not LL(1)?

- Consider the string: int + ( int ) + ( int )

## The Idea

- LR parsing *reduces* a string to the start symbol by inverting productions:

str w input string of terminals

repeat

- Identify $\beta$ in str such that $A \rightarrow \beta$ is a production (i.e., str = $\alpha\ \beta\ \gamma$)
- Replace $\beta$ by $A$ in str (i.e., str w = $\alpha\ A\ \gamma$)

until str = $S$ (the start symbol)

OR all possibilities are exhausted



## A Bottom-up Parse in Detail (1)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)

int + ( int ) + ( int )



## A Bottom-up Parse in Detail (2)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)

E + (int) + (int)

E

|

int + ( int ) + ( int )



## A Bottom-up Parse in Detail (3)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)

E + (int) + (int)

E + (E) + (int)

E E

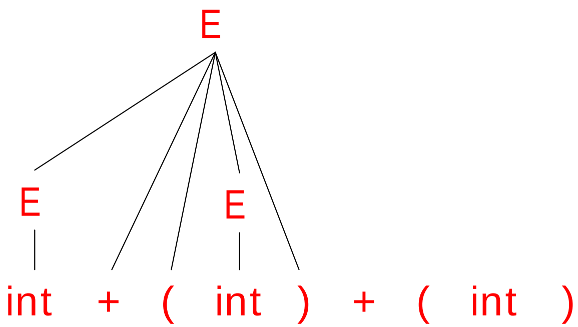
| |

int + ( int ) + ( int )

## A Bottom-up Parse in Detail (4)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)
E + (int) + (int)
E + (E) + (int)
E + (int)

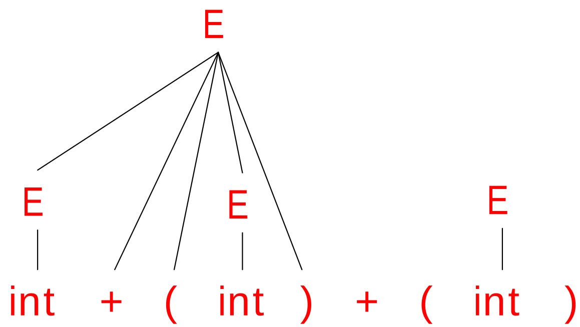





## A Bottom-up Parse in Detail (5)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)
E + (int) + (int)
E + (E) + (int)
E + (int)
E + (E)



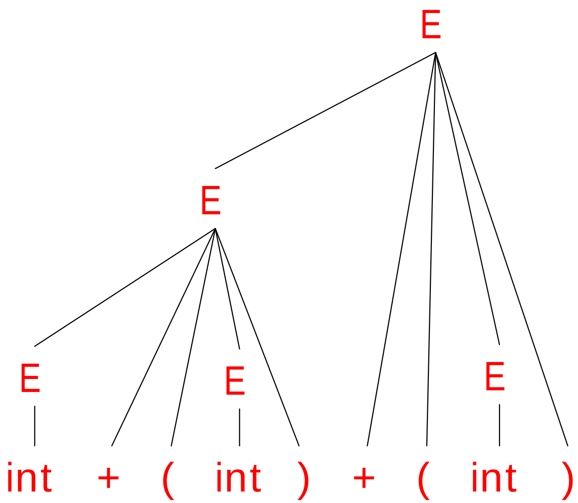



## A Bottom-up Parse in Detail (6)

$E \rightarrow E + ( E ) \mid int$

int + (int) + (int)
E + (int) + (int)
E + (E) + (int)
E + (int)
E + (E)
E

A rightmost derivation in reverse






## Important Fact #1 about Bottom-up Parsing

***An LR parser traces a rightmost derivation in reverse***

## Where Do Reductions Happen

Fact #1 has an interesting consequence:

- Let $\alpha\beta\gamma$ be a step of a bottom-up parse
- Assume the next reduction is by using $A \rightarrow \beta$
- Then $\gamma$ is a string of terminals

Why?

Because $\alpha A\gamma \rightarrow \alpha\beta\gamma$ is a step in a right-most derivation



## Notation

- Idea: Split string into two substrings
  - Right substring is as yet unexamined by parsing (a string of terminals)
  - Left substring has terminals and non-terminals
- The dividing point is marked by a I
  - The I is not part of the string
- Initially, all input is unexamined: $\mathrm{I} x_1 x_2 \ldots x_n$



## Shift-Reduce Parsing

Bottom-up parsing uses only two kinds of actions:

*Shift*

*Reduce*



## Shift

*Shift:* Move I one place to the right

- Shifts a terminal to the left string

$$E + (\mathrm{I}\ int\ ) \Rightarrow E + (int\ \mathrm{I}\ )$$

In general:

$$ABC\ \mathrm{I}\ xyz \Rightarrow ABCx\ \mathrm{I}\ yz$$

## Reduce

*Reduce:* Apply an inverse production at the right end of the left string

- If $E \rightarrow E + ( E )$ is a production, then

$$E + ( \underline{E + ( E )} \mid ) \Rightarrow E + ( \underline{E} \mid )$$

In general, given $A \rightarrow xy$, then:

$$Cbxy \mid ijk \Rightarrow CbA \mid ijk$$



## Shift-Reduce Example

$E \rightarrow E + ( E ) \mid int$

| | |
|---|---|
| \| int + (int) + (int)$ | shift |

int + ( int ) + ( int )

↑

## Shift-Reduce Example

$E \rightarrow E + ( E ) \mid int$

| | |
|---|---|
| \| int + (int) + (int)$ | shift |
| int \| + (int) + (int)$ | reduce $E \rightarrow int$ |

int + ( int ) + ( int )

↑

## Shift-Reduce Example

$E \rightarrow E + ( E ) \mid int$

| | |
|---|---|
| \| int + (int) + (int)$ | shift |
| int \| + (int) + (int)$ | reduce $E \rightarrow int$ |
| E \| + (int) + (int)$ | shift 3 times |

E

int + ( int ) + ( int )

↑

## Shift-Reduce Example

$E \rightarrow E + ( E ) \mid int$

| | |
|---|---|
| \| int + (int) + (int)$ | shift |
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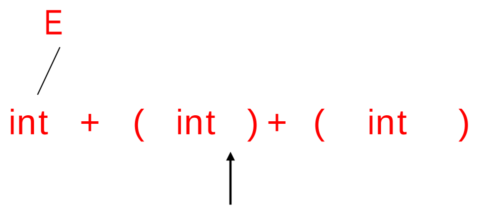


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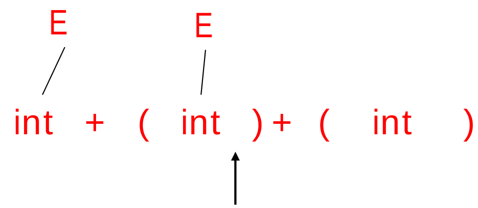
| E + (int \| ) + (int)$ | reduce $E \rightarrow int$ |
| E + (E \| ) + (int)$ | shift |

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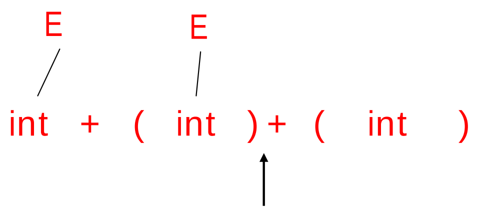
$E \rightarrow E + ( E ) \mid int$

| | |
|---|---|
| \| int + (int) + (int)$ | shift |
| int \| + (int) + (int)$ | reduce $E \rightarrow int$ |
| E \| + (int) + (int)$ | shift 3 times |
| E + (int \| ) + (int)$ | reduce $E \rightarrow int$ |
| E + (E \| ) + (int)$ | shift |
| E + (E) \| + (int)$ | reduce $E \rightarrow E + (E)$ |

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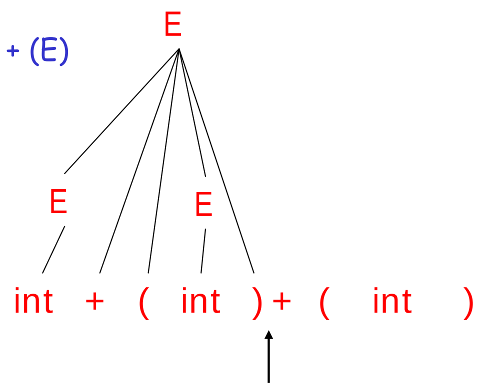
$E \rightarrow E + ( E ) \mid int$

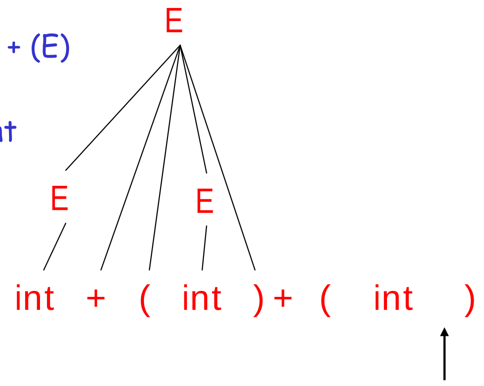
| | |
|---|---|
| \| int + (int) + (int)$ | shift |
| int \| + (int) + (int)$ | reduce $E \rightarrow int$ |
| E \| + (int) + (int)$ | shift 3 times |
| E + (int \| ) + (int)$ | reduce $E \rightarrow int$ |
| E + (E \| ) + (int)$ | shift |
| E + (E) \| + (int)$ | reduce $E \rightarrow E + (E)$ |
| E \| + (int)$ | shift 3 times |



## Shift-Reduce Example

$E \to E + ( E ) \mid int$

| | |
|---|---|
| $\vert$ int + (int) + (int)$ | shift |
| int $\vert$ + (int) + (int)$ | reduce E $\to$ int |
| E $\vert$ + (int) + (int)$ | shift 3 times |
| E + (int $\vert$ ) + (int)$ | reduce E $\to$ int |
| E + (E $\vert$ ) + (int)$ | shift |
| E + (E) $\vert$ + (int)$ | reduce E $\to$ E + (E) |
| E $\vert$ + (int)$ | shift 3 times |
| E + (int $\vert$ )$ | reduce E $\to$ int |

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| E + (E $\vert$ ) + (int)$ | shift |
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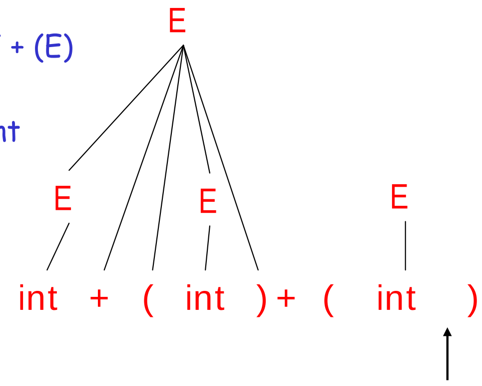
| E $\vert$ + (int)$ | shift 3 times |
| E + (int $\vert$ )$ | reduce E $\to$ int |
| E + (E $\vert$ )$ | shift |




## Shift-Reduce Example

$E \to E + ( E ) \mid int$

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| E + (E $\vert$ ) + (int)$ | shift |
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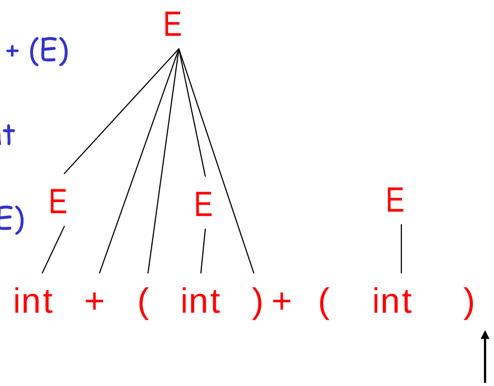
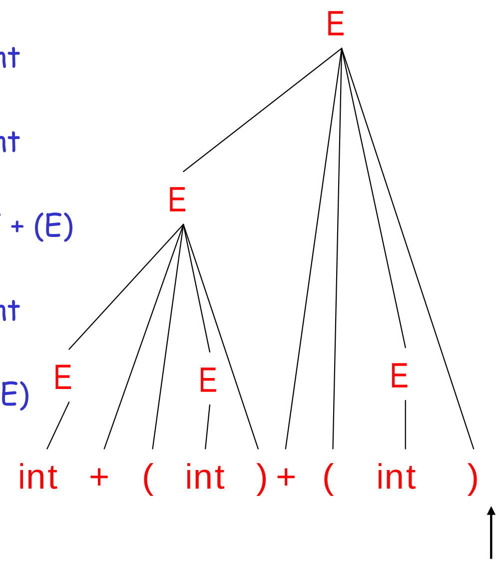
| E + (int $\vert$ )$ | reduce E $\to$ int |
| E + (E $\vert$ )$ | shift |
| E + (E) $\vert$ $ | reduce E $\to$ E + (E) |




## Shift-Reduce Example

| | |
|---|---|
| $\vert$ int + (int) + (int)$ | shift |
| int $\vert$ + (int) + (int)$ | reduce E $\to$ int |
| E $\vert$ + (int) + (int)$ | shift 3 times |
| E + (int $\vert$ ) + (int)$ | reduce E $\to$ int |
| E + (E $\vert$ ) + (int)$ | shift |
| E + (E) $\vert$ + (int)$ | reduce E $\to$ E + (E) |
| E $\vert$ + (int)$ | shift 3 times |
| E + (int $\vert$ )$ | reduce E $\to$ int |
| E + (E $\vert$ )$ | shift |
| E + (E) $\vert$ $ | reduce E $\to$ E + (E) |
| E $\vert$ $ | accept |



## The Stack

- Left string can be implemented by a stack
  - Top of the stack is the |
- Shift pushes a terminal on the stack
- Reduce pops 0 or more symbols off of the stack (production RHS) and pushes a non-terminal on the stack (production LHS)



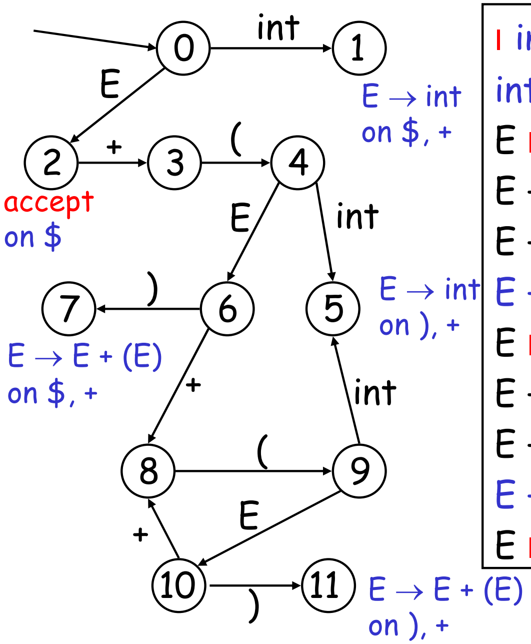
## Key Question: To Shift or to Reduce?

**<u>Idea</u>**: use a finite automaton (DFA) to decide when to shift or reduce
- The input is the stack
- The language consists of terminals and non-terminals

- We run the DFA on the stack and we examine the resulting state X and the token tok after |
  - If X has a transition labeled tok then <u>shift</u>
  - If X is labeled with "$A \to \beta$ on tok" then <u>reduce</u>



## LR(1) Parsing: An Example



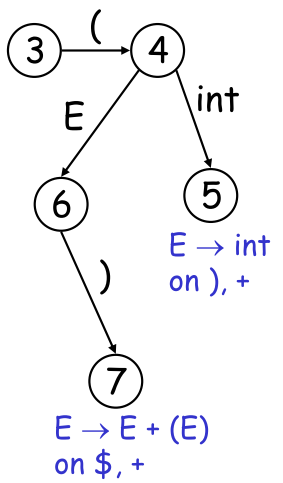

| | |
|---|---|
| \| int + (int) + (int)$ | shift |
| int \| + (int) + (int)$ | E → int |
| E \| + (int) + (int)$ | shift(x3) |
| E + (int \| ) + (int)$ | E → int |
| E + (E \| ) + (int)$ | shift |
| E + (E) \| + (int)$ | E → E+(E) |
| E \| + (int)$ | shift (x3) |
| E + (int \| )$ | E → int |
| E + (E \| )$ | shift |
| E + (E) \| $ | E → E+(E) |
| E \| $ | accept |

## Representing the DFA

- Parsers represent the DFA as a 2D table
  - Recall table-driven lexical analysis
- Lines correspond to DFA states
- Columns correspond to terminals and non-terminals
- Typically columns are split into:
  - Those for terminals: action table
  - Those for non-terminals: goto table

## Representing the DFA: Example

- The table for a fragment of our DFA:




| | int | + | ( | ) | $ | E |
|---|---|---|---|---|---|---|
| … | | | | | | |
| 3 | | | s4 | | | |
| 4 | s5 | | | | | g6 |
| 5 | | $r_{E \to int}$ | | $r_{E \to int}$ | | |
| 6 | s8 | | s7 | | | |
| 7 | | $r_{E \to E+(E)}$ | | | $r_{E \to E+(E)}$ | |
| … | | | | | | |



## The LR Parsing Algorithm

- After a shift or reduce action we rerun the DFA on the entire stack
  - This is wasteful, since most of the work is repeated
- Remember for each stack element on which state it brings the DFA
- LR parser maintains a stack

$\langle sym_1, state_1 \rangle \ldots \langle sym_n, state_n \rangle$

$state_k$ is the final state of the DFA on $sym_1 \ldots sym_k$



## The LR Parsing Algorithm

```
let I = w$ be initial input
let j = 0
let DFA state 0 be the start state
let stack = ⟨ dummy, 0 ⟩
  repeat
    case action[top_state(stack), I[j]] of
      shift k: push ⟨ I[j++], k ⟩
      reduce X → A:
         pop |A| pairs,
         push ⟨X, Goto[top_state(stack),X] ⟩
      accept: halt normally
      error: halt and report error
```



## LR Parsers

- Can be used to parse more grammars than LL
- Most programming languages grammars are LR
- LR Parsers can be described as a simple table
- There are tools for building the table
- How is the table constructed?