Computing Scaling Relations by Solving the Full Stokes Equations

Assessing the validity of the SIA

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1. Goal

The Shallow Ice Approximation (SIA) is derived under two assumptions:

1. That the ice sheet is shallow, i.e. $\varepsilon=[H]/[L] << 1$,
2. That the vertical shear stress and horizontal velocity dominates, i.e. **that certain scaling relations hold**.

However, in literature there are different scaling relations. **By numerical experiments, using Elmer, we investigate what scaling relations are valid**. We compare our results to scaling relations in **Baral et al. (2001), Blatter (1995)** and **Schoof & Hindmarsh (2010)**.

3b. Method - Boundary layer

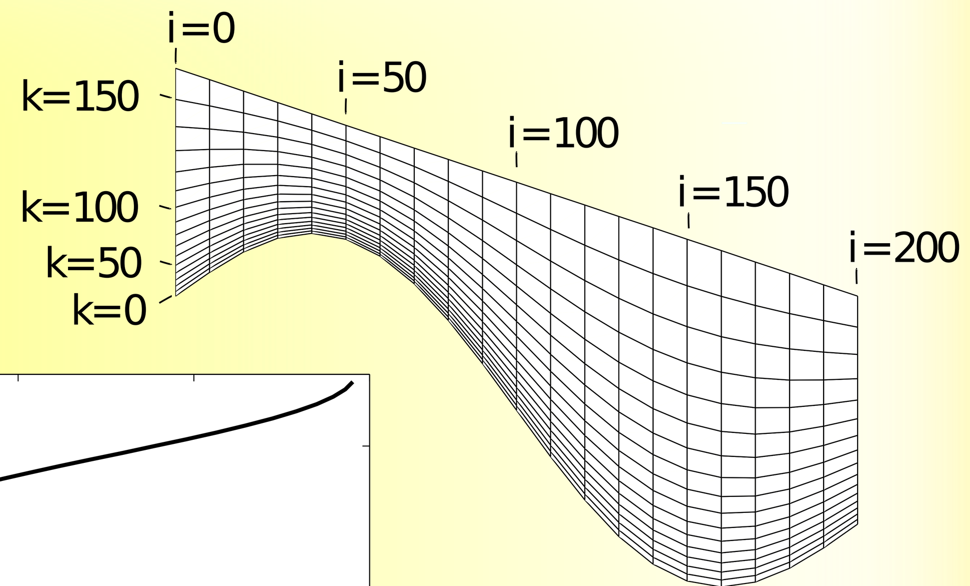

Figure 3. Grid with horizontal layers, k.

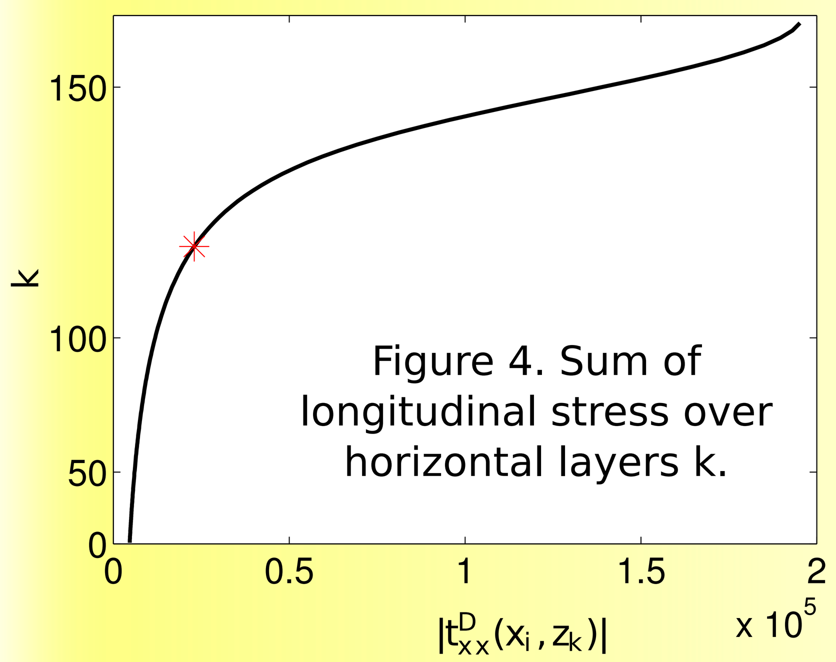

Figure 4. Sum of longitudinal stress over horizontal layers k.

To measure the boundary layer thickness numerically, the longitudinal stress is summarized over a horizontal layer, and the boundary layer border is defined to be at **10 % of the maximum value** (red star in Figure 4). **Separate scaling relations inside and outside the boundary layer are then computed.**

2. Theory - Scaling Relations

Baral et al assumes that

$t_{xz} \sim \rho g[H]\varepsilon^1$, $t_{xx} \sim \rho g[H]\varepsilon^2$

$v_z/v_x \sim \varepsilon$

While Blatter uses

$t_{xz} \sim \rho g[H]\varepsilon^1$, $t_{xx} \sim \rho g[H]\varepsilon^1$

$v_x \sim A[H](\rho g[H])^3\varepsilon^3$, $v_z \sim A[H](\rho g[H])^3\varepsilon^4$

Johnson & McMeeking and Schoof & Hindmarsh states that there is a high viscosity (coinciding with higher longitudinal stress) **boundary layer** near the ice surface with thickness varying as $\varepsilon^{1/3}$, and that in this boundary layer the variables need to be **rescaled**. The scalings in **Schoof & Hindmarsh applied to our problem are:**

Outside the boundary layer, Ω_o

$t_{xz} \sim \rho g[H]\varepsilon^1$, $t_{xx} \sim \rho g[H]\varepsilon^2$

$v_x \sim A[H](\rho g[H])^3\varepsilon^3$, $v_z \sim A[H](\rho g[H])^3\varepsilon^4$

In the boundary layer, Ω_i

$t_{xz} \sim \rho g[H]\varepsilon^{4/3}$, $t_{xx} \sim \rho g[H]\varepsilon^{4/3}$

$v_x \sim A[H](\rho g[H])^3\varepsilon^3$, $v_z \sim A[H](\rho g[H])^3\varepsilon^4$

4. Results

We have varied geometrical parameters. For a angle α independent of ε the scaling relations do not agree with any theory. For α=arctan(ε)° we vary the bump amplitude:

Table 1. **α=arctan(ε)°, bump amplitude**

Variable	Ω	Ω_o	Ω_i
H_{bl}	$2.7\rho g[H]\varepsilon^{0.26}$	-	-
t_{xz}	$0.61\rho g[H]\varepsilon^{1.0}$	$1.1\rho g[H]\varepsilon^{1.1}$	$1.6\rho g[H]\varepsilon^{1.3}$
t_{xx}	$2.8\rho g[H]\varepsilon^{1.5}$	$1.2\rho g[H]\varepsilon^{1.7}$	$1.7\rho g[H]\varepsilon^{1.4}$
v_x	$1.1A[H](\rho g[H])^3\varepsilon^{3.0}$	$0.53A[H](\rho g[H])3\varepsilon^{2.9}$	$1.2A[H](\rho g[H])3\varepsilon^{3.0}$
v_z	$4.9A[H](\rho g[H])^3\varepsilon^{4.0}$	$0.93A[H](\rho g[H])^3\varepsilon^{3.9}$	$4.4A[H](\rho g[H])^3\varepsilon^{4.0}$

Table 2. **α=arctan(ε)°, bump amplitude**

Variable	Ω	Ω_o	Ω_i
H_{bl}	$2.2\rho g[H]\varepsilon^{0.31}$	-	-
t_{xz}	$0.58\rho g[H]\varepsilon^{1.0}$	$0.86\rho g[H]\varepsilon^{1.0}$	$1.3\rho g[H]\varepsilon^{1.3}$
t_{xx}	$1.12\rho g[H]\varepsilon^{1.5}$	$0.32\rho g[H]\varepsilon^{1.6}$	$0.77\rho g[H]\varepsilon^{1.4}$
vx	$0.44A[H](\rho g[H])^3\varepsilon^{3.0}$	$0.35A[H](\rho g[H])^3\varepsilon^{3.0}$	$0.58A[H](\rho g[H])^3\varepsilon^{3.0}$
v_z	$0.71A[H](\rho g[H])3\varepsilon^{4.0}$	$0.39A[H](\rho g[H])3\varepsilon^{3.9}$	$0.89A[H](\rho g[H])3\varepsilon^{4.0}$

The boundary layer remains even when lowering the bump amplitude to 0.05[H]. At zero bump amplitude there is no boundary layer.

3a. Method - calculating scalings

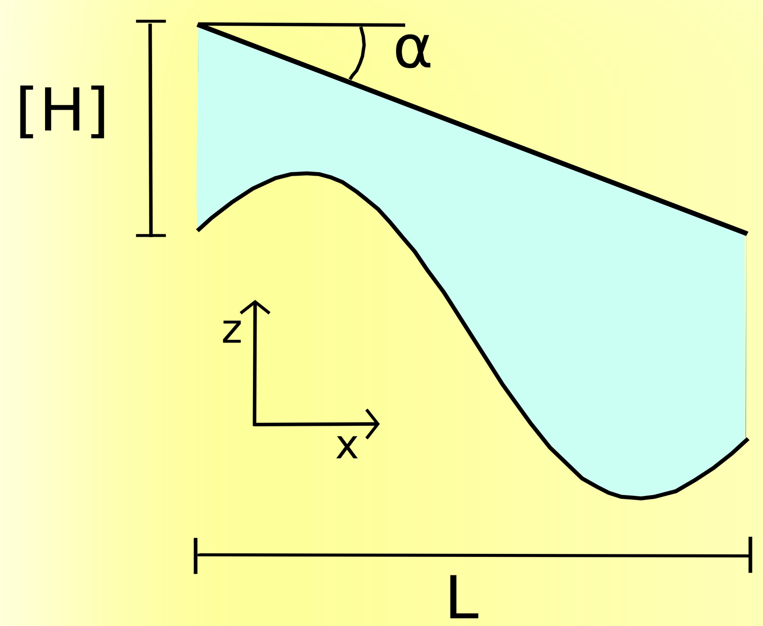

Figure 1. Model problem. [H] is constant at 1 km, while L is varied.

Elmer is run for the ISMIP-HOM inspired problem in Figure 1. The length, L, of the domain is varied between 10 km and 10 240 km, which means that **ε is varied**.

For each ε, the L_2-norm of the field variables is computed. **Doing a polynomial fit for the norm for the smallest ε (Figure 2), scaling relations are obtained.**

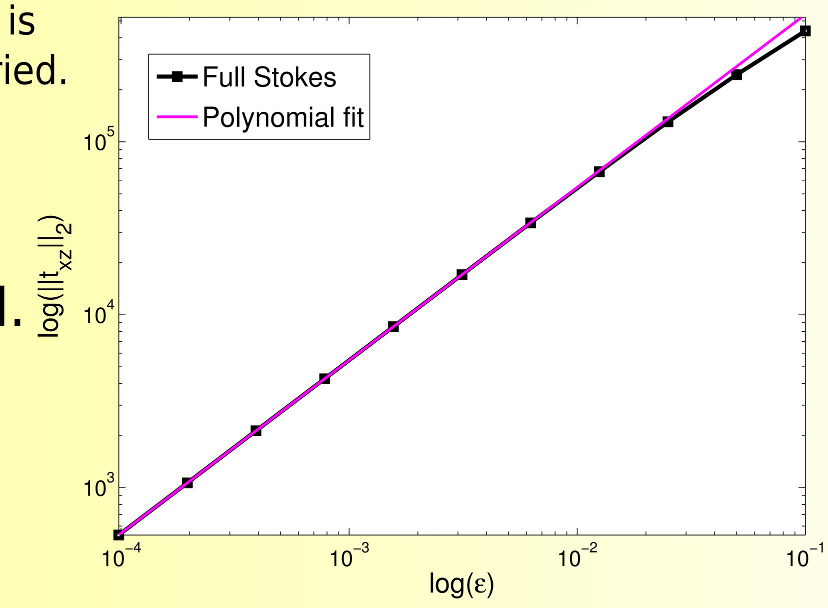

Figure 2. The L_2- norm of the vertical shear stress t_{xz} for different ε (black line). The polynomial fit (pink line) agrees well.

5. Conclusions

- There is a thick high viscosity boundary layer near the ice surface, which depends on ε, **agreeing fairly well with theory in Johnson & McMeeking (1984).**

- The **boundary layer** develops **immediately as bumps are introduced** at the bed.

- The scalings in Baral et al., which the SIA is derived from, **do not take the boundary layer into account**. This matters when going to **second order** (SOSIA).

- The **scalings in Schoof & Hindmarsh are in good agreement** with our results, **except for the longitudinal stress t_{xx}**, which does not behave as ε^2 outside the boundary layer.

- The scaling relations behind the SIA do not hold for slope angles independent of the aspect ratio ε.

References: [1]: Baral et al. (2001) *Asymptotic theories of large-scale motion, temperature and moisture distribution in land-based polythermal ice sheets: A critical review and new developments*, Applied Mechanics Reviews, 54(3), 215-256.

[2]: Blatter (1995) *Velocity and Stress Fields in Grounded Glaciers: a Simple Algorithm for Including Deviatoric Stress Gradients*, Journal of Glaciology, 41(138),333-334.

[3]: Schoof and Hindmarsh (2010)Thin-Film Flows with Wall Slip: An Asymptotic Analysis of Higher Order Glacier Flow Models. Quarterly Journal of Mechanics and Applied Mathematics, 63, 73-11.

[4]: Johnson and McMeeking *(1984) Near-surface flow in glaciers obeying Glen's Law,* Quarterly Journal of Mechanics and Applied Mathematics, 37(2), 273-291.

[5]: Pattyn et al. (2008) Benchmark experiments for higher-order and full-Stokes ice sheet models (ISMIP-HOM), the Cryosphere,2,95-108.