Second-order Shallow Ice Approximation with Non-linear Rheology: Exploring Validity by Performing Numerical Experiments

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1. Goal

- By numerical experiments we investigate if the shallow ice approximation assumptions are valid. We have used the full Stokes code **Elmer.**

- We compute the accuracy of SIA and SOSIA (Second Order Shallow Ice Approximation) [1] **numerically, and illustrate how the accuracy depends on the aspect ratio ε.** We have implemented the SOSIA in a MATLAB-code to perform experiments and used Elmer for comparison.

- We investigate how sensitive SOSIA is to the choice of finite viscosity law, that is, using the creep response function $\mathbf{f(\sigma)=\sigma^2+k^2}$ instead of $f(\sigma)=\sigma^2$ to avoid singularites, where k is a constant. We use both **numerical results and an analytical solution** for the SOSIA that we have constructed for the problem in Figure 1.

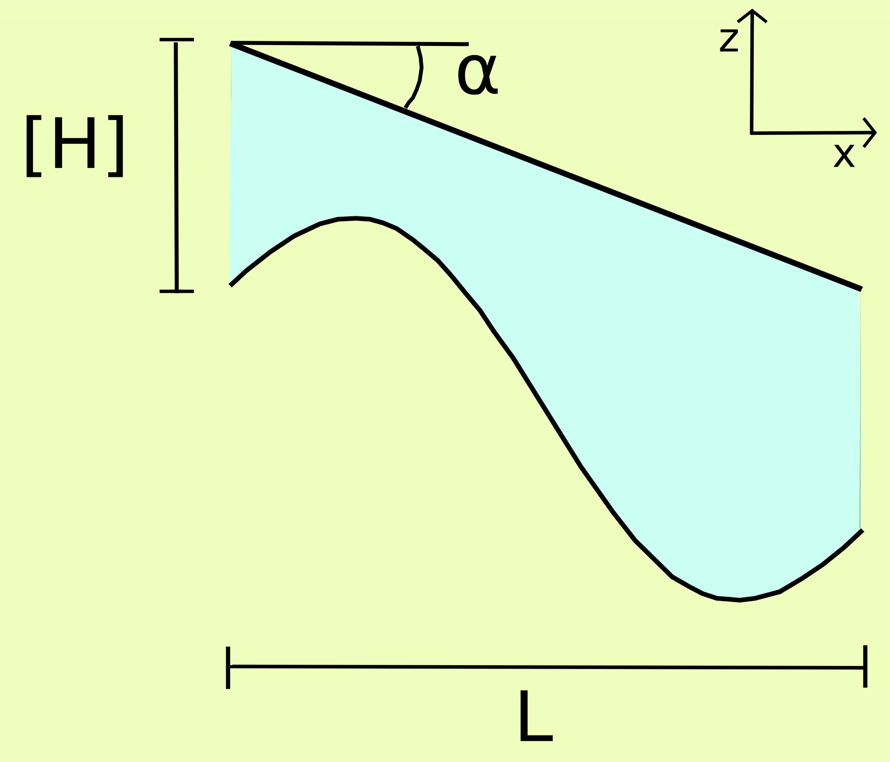

Figure 1. Model problem. [H] is constant at 1 km, while L is varied.

A modified **ISMIP-HOM experiment B** [3] is used. For the shallow ice theory to hold, **the angle should be α = arctan(ε)**, where ε is the aspect ratio, ε=[H]/L.

2. Numerical Test of the Scaling

Elmer is run for the problem in Figure 1. The length, L, of the domain is varied between 10 km and 10 240 km, which means that **ε is varied**.

For each ε, the **$\mathbf{L_2}$-norm of the field variables** is computed. Doing a **polynomial fit for the norm** for the smallest ε (Figure 2), **scaling relations are obtained** (table below).

Variable	Numerical	Theory
t_{xz}	$0.61\rho g[H]\varepsilon^{1.0}$	$\rho g[H]\varepsilon^{1}$ [1]
t_{xx}	$2.8\rho g[H]\varepsilon^{1.5}$	$\rho g[H]\varepsilon^{2}$ [1]
v_x	$2.4\varepsilon^{3.0}$	$2.25\varepsilon^{3}$ [2]

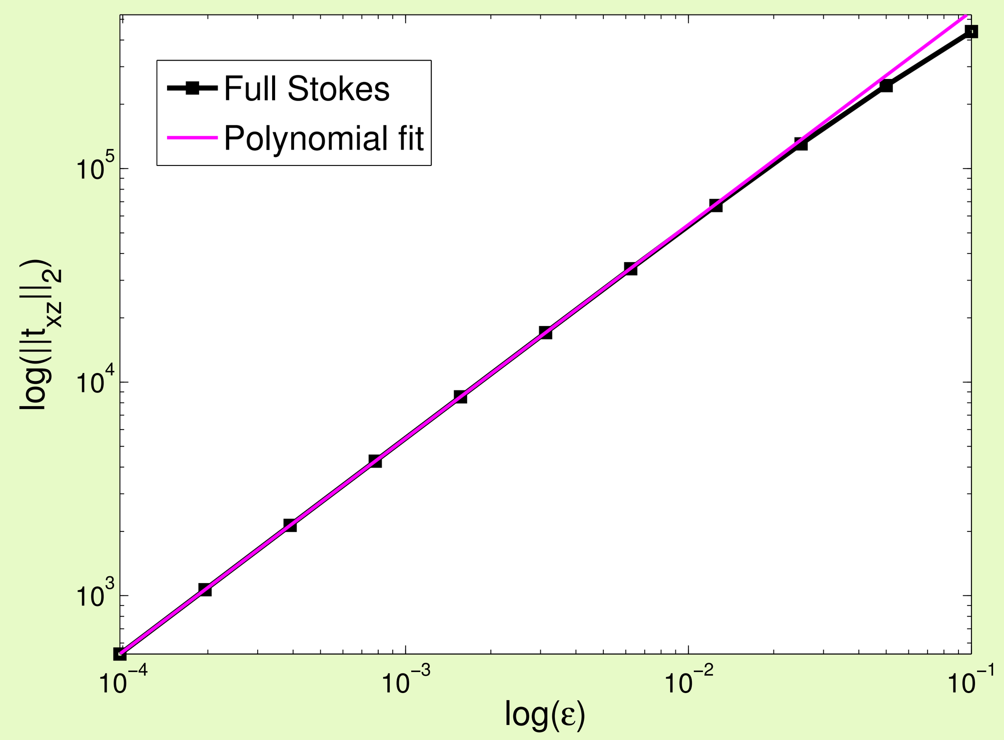

Figure 2. The L_2- norm of the vertical shear stress t_{xz} for different ε (black line). The polynomial fit (pink line) agrees well.

The scalings for each variable agree well with theory, except for the normal deviatoric stress, $\mathbf{t_{xx}}$. When decreasing the amplitude of the bumps, the scaling of t_{xx} approaches theory. In line with theory in [4], a boundary layer exists. We find that the scalings are better, but not good, if the boundary layer is excluded.

3. Accuracy of SOSIA

We have run SIA and SOSIA for varying ε, and compared to Elmer results to compute the accuracy.

From the analytical solutions we see that **k should be Cρg[H]ε**. A too small a C leads to a dip in vertical shear stress and velocity at the trough. A too big a C gives a large extra zeroth order term, leading to a too high overall velocity (Figure 3). **These behaviours are explained by the analytical solutions**.

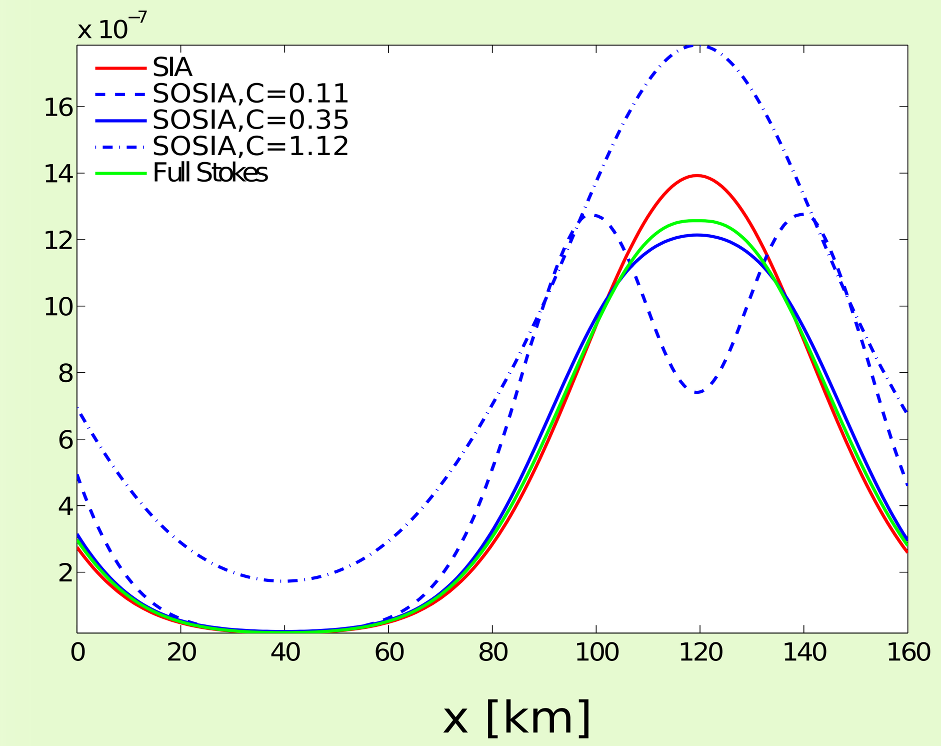

Figure 3. Velocity in x-direction at the surface, L=160 km.

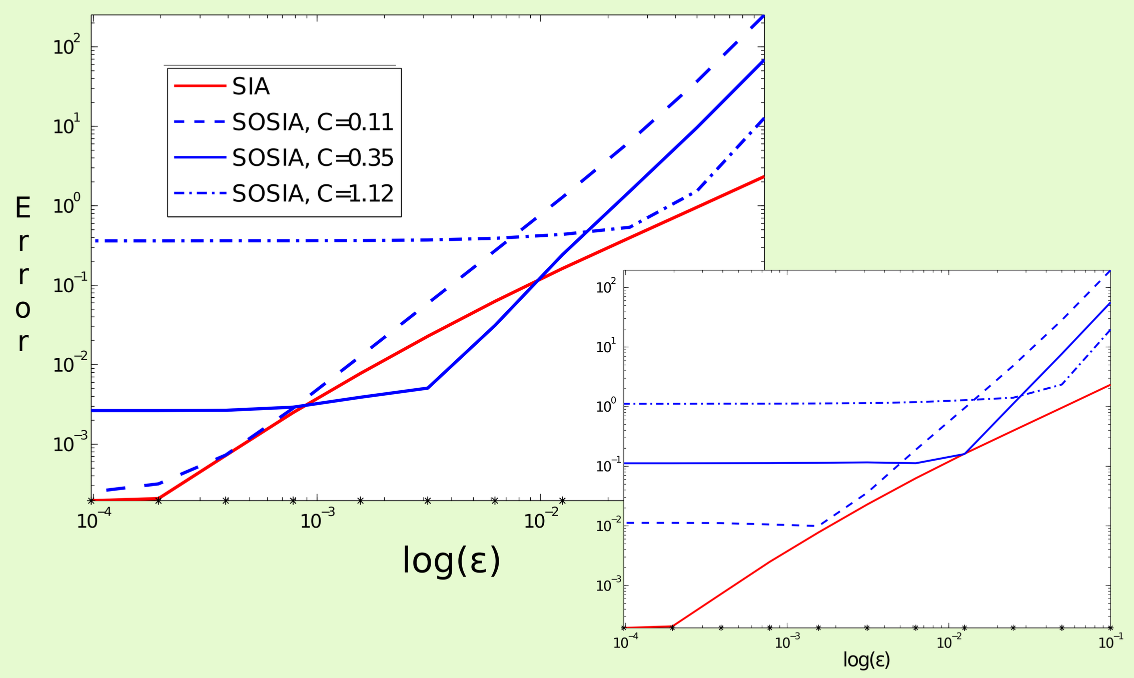

Figure 4. SOSIA and SIA velocity error for different ε. In the small picture, k is added everywhere. In the large picture, k is used as a lower limit of the effective stress σ.

Adding **k everywhere** leads to a SOSIA error that is never smaller than the SIA error. Using **k as a lower limit** on the effective stress, the accuracy is improved.

The error of SOSIA is very large for large ε.

4. Iterating SOSIA

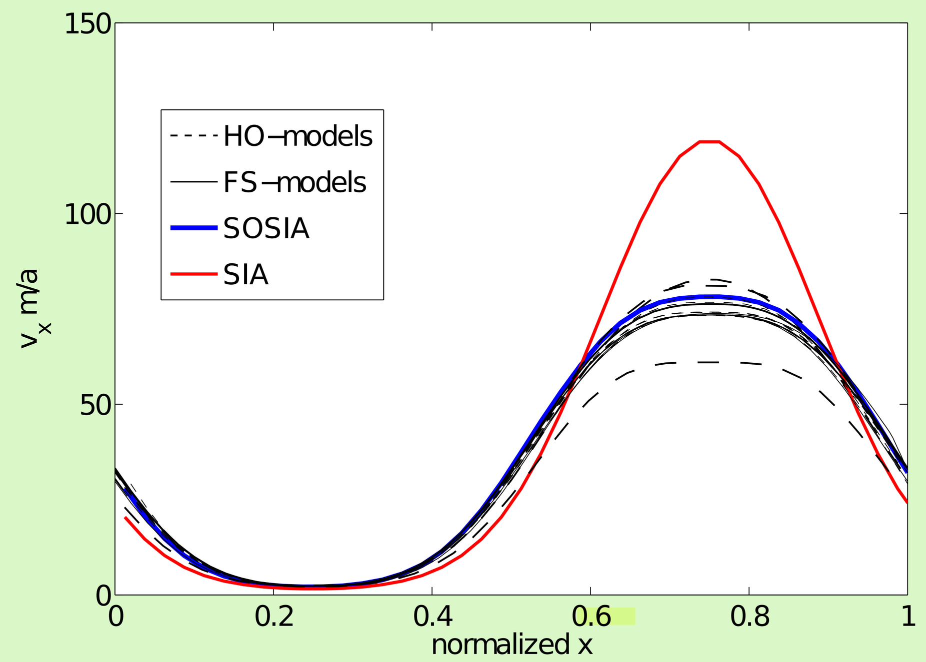

Figure 5. Velocity in x-direction at the surface, L=40 km. SOSIA is iterated.

When iterating SOSIA (fixed point iterations of the velocity)**, it converges to a solution that is close to the full Stokes solution,** even for **larger ε** and a **non-optimized k** that is added everywhere.

5. Conclusions

- The shallow ice scalings are correct for all field variables for the problem in Figure 1, except for the normal deviatoric stress, $\mathbf{t_{xx}}$.

- The SOSIA is sensitive to the choice of the parameter k in the finite viscosity law. Further, k should not be added everywhere, but should be used as a lower limit of the effective stress**.**

- For small ε, the SOSIA is more accurate than the SIA, but for large ε, the accuracy of SOSIA is far worse. Due to its sensitivity to k, bump amplitude and ε, **it is risky to use SOSIA, without some kind of automatic error control or iterations.**

References: [1]: Baral et al. (2001) *Asymptotic thoeries of large-scale motion, temperature and moisture distribution in land-based polythermal ice sheets: A critical review and new developments*, Applied Mechanics Reviews, 54(3), 215-256.
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[3]: Pattyn et al. (2008) *Benchmark experiments for higher-order and full-Stokes ice sheet models (ISMIP-HOM),* the Cryosphere,2,95-108.
[4]: Johnson and McMeeking *(1984) Near-surface flow in glaciers obeying Glen's Law,* Quarterlu Journal of Mechanics and Applied Mathematics, 37(2), 273-291.