The one dimensional wave equation

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Find all solutions to

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

First define a new coordinate (ξ, η), where

$$\xi = x - ct, \qquad \eta = x + ct$$

This means we are now looking for a function $u(\xi(x,t), \eta(x,t))$. First we need to find u_{xx} and u_{tt}.

We begin with u_{xx}. Apply the chain rule:

$$\frac{\partial}{\partial x}\left(u(\xi(x,t), \eta(x,t))\right) =$$

$$\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial x} = \left[\frac{\partial \xi}{\partial x} = 1 = \frac{\partial \eta}{\partial x}\right] = \frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}$$

And again:

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial \xi} + \frac{\partial u}{\partial \eta}\right) = \frac{\partial}{\partial x}\frac{\partial u}{\partial \xi} + \frac{\partial}{\partial x}\frac{\partial u}{\partial \eta} =$$

$$\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x} = \left[\frac{\partial \xi}{\partial x} = 1 = \frac{\partial \eta}{\partial x}\right] =$$

$$\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

Now u_{tt}:

$$\frac{\partial}{\partial t}\left(u(\xi(x,t),\eta(x,t))\right) =$$

$$\frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial t} + \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial t} = \left[\frac{\partial \xi}{\partial x} = -c, \frac{\partial \eta}{\partial x} = c\right] = -c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}$$

Differentiate again:

$$\frac{\partial}{\partial t}\left(-c\frac{\partial u}{\partial \xi} + c\frac{\partial u}{\partial \eta}\right) = -c\frac{\partial}{\partial x}\frac{\partial u}{\partial \xi} + c\frac{\partial}{\partial x}\frac{\partial u}{\partial \eta} =$$

$$-c\left(\frac{\partial^2 u}{\partial \xi^2}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \xi \partial \eta}\frac{\partial \eta}{\partial x}\right) + c\left(\frac{\partial^2 u}{\partial \eta \partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial^2 u}{\partial \eta^2}\frac{\partial \eta}{\partial x}\right) =$$

$$\left[\frac{\partial \xi}{\partial x} = -c, \frac{\partial \eta}{\partial x} = c\right] =$$

$$-c\left(-c\frac{\partial^2 u}{\partial \xi^2} + c\frac{\partial^2 u}{\partial \xi \partial \eta}\right) + c\left(-c\frac{\partial^2 u}{\partial \eta \partial \xi} + c\frac{\partial^2 u}{\partial \eta^2}\right) =$$

$$c^2\frac{\partial^2 u}{\partial \xi^2} - c^2\frac{\partial^2 u}{\partial \xi \partial \eta} - c^2\frac{\partial^2 u}{\partial \eta \partial \xi} + c^2\frac{\partial^2 u}{\partial \eta^2} =$$

$$c^2\frac{\partial^2 u}{\partial \xi^2} - 2c^2\frac{\partial^2 u}{\partial \xi \partial \eta} + c^2\frac{\partial^2 u}{\partial \eta^2} =$$

$$c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)$$

So,

$$u_{xx} = \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}$$

$$u_{tt} = \frac{\partial^2 u}{\partial x^2} = c^2\left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2}\right)$$

Now insert these expressions into the wave equation.

$$c^2 \frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$$

$$c^2 \left(\frac{\partial^2 u}{\partial \xi^2} + 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right) = c^2 \left(\frac{\partial^2 u}{\partial \xi^2} - 2\frac{\partial^2 u}{\partial \xi \partial \eta} + \frac{\partial^2 u}{\partial \eta^2} \right)$$

$$c^2 \frac{\partial^2 u}{\partial \xi^2} + 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2} = c^2 \frac{\partial^2 u}{\partial \xi^2} - 2c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} + c^2 \frac{\partial^2 u}{\partial \eta^2}$$

After simplifying this expression, we arrive at

$$4c^2 \frac{\partial^2 u}{\partial \xi \partial \eta} = 0$$

which can further be simplified to

$$\frac{\partial^2 u}{\partial \xi \partial \eta} = 0 \iff \frac{\partial}{\partial \xi} \left(\frac{\partial u}{\partial \eta} \right) = 0$$

Now we integrate the expression

$$\frac{\partial^2 u}{\partial\xi\partial\eta} = 0$$

with regard to ξ:

$$\int \frac{\partial^2 u}{\partial\xi\partial\eta} \; d\xi = \int 0 \; d\xi$$

which yields

$$\frac{\partial u}{\partial\eta} = h(\eta)$$

where $h(\eta)$ is constant with regard to ξ. If ψ is an antiderivate of $h(\eta)$, another integration will yield a constant function, with regard to η, which we call $\varphi(\xi)$:

$$u(\xi, \eta) = \int h(\eta) \; d\eta + \varphi(\xi) = \varphi(\xi) + \psi(\eta)$$

Finally, we resubsitute with the original (x, t) coordinates:

$$u(x, t) = \varphi(x - ct) + \psi(x + ct)$$

Solutions to the one dimensional wave equation must be in this form, and $\varphi, \psi \in C^2$ or higher.