# Adaptive discontinuous Galerkin multiscale methods for elliptic problems:

## Energy norm a posteriori error estimate

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# Outline



## Papers

- D. Elfverson, E. Georgoulis and A. Målqvist, Adaptive discontinuous Galerkin multiscale method: Energy norm a posteriori error estimate.

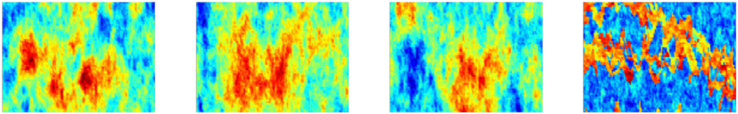
# Model problem

## Poisson's equation

Given a polygonal domain $\Omega \subset \mathbf{R}^d$. We want to find $u$ such that

$$
\begin{aligned}
-\nabla \cdot \alpha \nabla u &= f \text{ i } \Omega, \\
n \cdot \nabla u &= 0 \text{ on } \partial\Omega,
\end{aligned}
$$

where $\alpha$ is bounded $0 < \alpha_0 \leq \alpha(x) \leq \alpha^0$ and $f \in L^2(\Omega)$. $L^2$ has the inner product $(u, v) = \int_\Omega uv \, dx$ and norm $\|u\|^2 = \int_\Omega u^2 \, dx$.



(a) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (b) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (c) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (d) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^6$

Figure: Permeabilities $\alpha$ projection in log scale.

# Weak formulation using a discontinuous Galerkin discretization (SIPG)

- Let $\Omega$ be subdivided into the partition $\mathcal{K} = \{K\}$ and $\Gamma^I$ be the union of all interior edges.
- Let also $\mathcal{V}^h$ be the space of all discontinuous piecewise linear polynomials.

The bilinear form and right hand side are defined as:

$$\begin{aligned} a(v,w) &= \sum_{k\in\mathcal{K}} (\alpha\nabla v, \nabla w)_K - \sum_{e\in\Gamma} (\mathbf{n}\cdot\{\alpha\nabla v\}, [w])_e \\ &\quad - \sum_{e\in\Gamma^I} (\mathbf{n}\cdot\{a\nabla w\}, [v])_{\partial K} + \sum_{e\in\Gamma^I} \frac{\sigma_e}{h_e}([v],[w])_e. \\ l(v) &= (f, v). \end{aligned}$$

The discontinuous Galerkin method reads: find $u_h \in \mathcal{V}^h$ s.t

$$a(u_h, v) = l(v) \text{ for all } v \in \mathcal{V}^h$$

# Why do we need to resolve the coefficients?

## Example with periodic coefficient

Consider Possion's equation with period coefficient $\alpha = \alpha(x/\epsilon)$. For the finite element method, we have (Hou-Wu-Cai),

$$||\sqrt{\alpha}\nabla(u - u_h)||_{L^2(\Omega)} \leq C\frac{H}{\epsilon}||f||_{L^2(\Omega)}$$

- Need $H \ll \epsilon$ for reliable results.
- To computational expensive to solve on a single mesh for many applications e.g flow in porous media and in composite materials.
- Want eliminate the $\epsilon$ dependence by using a multiscale method (Målqvist-Peterseim).

# Framework for Multiscale methods

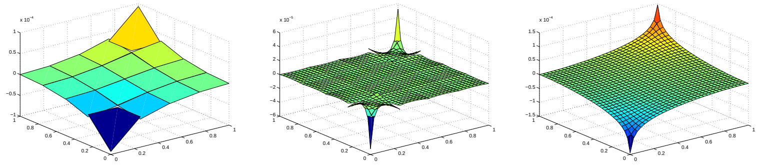
The problem is split into one coarse and fine scale contribution $\mathcal{V}^h = \mathcal{V}_c \oplus \mathcal{V}_f^h$.

- Let subdivide $\Omega$ into a coarse mesh $\mathcal{K}_c = \{K_c\}$.
- $\mathcal{V}_c = span\{\phi_i\} = \mathcal{I}_c \mathcal{V}^h$ and $\mathcal{V}_f^h = \{v \in \mathcal{V}^h : \ \mathcal{I}_c v = 0\}$, where $\mathcal{I}_c : \mathcal{V}^h \to \mathcal{V}_c$ is a inclusion operator.
- Define the map $\mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f^h$ as $a(\mathcal{T} v_c, v_f) = -a(v_c, v_f)$.

Split $U = U_c + \mathcal{T} U_c + U_f$ and $v = v_c + v_f$ where $u_c \in \mathcal{V}_c$, $v_f \in \mathcal{V}_f^h$.

$$a(U_c + \mathcal{T} U_c + U_f, v_c + v_f) = l(v_c + v_f)$$

for all $v_c \in \mathcal{V}_c$ and $v_f \in \mathcal{V}_f^h$



(a) $U_C$ (b) $\mathcal{T} U_c + U_f$ (c) $U = U_c + \mathcal{T} U_c + U_f$

## Fine scale

Let $v_c = 0$ to get the fine scale equations

$$a(\mathcal{T}U_c + U_f, v_f) = l(v_f) - a(U_c, v_f),$$

split into two equations

$$a(U_f, v_f) = l(v_f) \quad \forall v_f \in \mathcal{V}_f^h,$$
$$a(\mathcal{T}U_c, v_f) = -a(U_c, v_f) \quad \forall v_f \in \mathcal{V}_f^h.$$

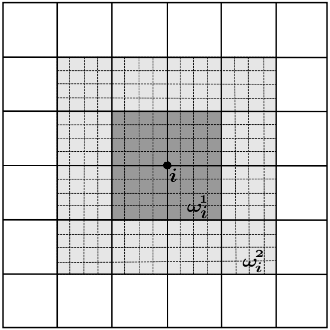
## Coarse scale

Let $v_f = 0$ on the coarse scale

$$a(U_c + \mathcal{T}U_c, v_c) = l(v_c) - a(U_f, v_c) \quad \forall v_c \in \mathcal{V}_c$$

# Approximation of $\mathcal{T} U_c$ and $U_f$

Because of the local behavior in $\mathcal{V}_f^h$, $\mathcal{T} U_c$ and $U_f$ can be solved on a subset $\omega_i^L$ instead of the whole domain $\Omega$ i.e. $V_f^h(\omega_i^L) \subset V_f^h$ (Målqvist-Peterseim).



# Multiscale method discretization

- Let $\mathcal{I}_c = P_c$ be the piecewise linear $L^2$-projection onto the coarse mesh.
- $\tilde{U}_f = \sum_{i\in\mathcal{N}} \tilde{U}_{f,i}$ where $\mathcal{N}$ is the number of nodes.
- $\mathcal{M}_i$ be all $j$ s.t $\phi_j = 1$ in node $i$.
- Let also $\Phi_i = \sum_{j\in\mathcal{M}_i} \phi_j$

## Fine scale equations

For all $i \in \mathcal{N}$: find $\tilde{\mathcal{T}}\phi_j \in \mathcal{V}_f^h(\omega_i^L)$ and $U_{f,i} \in \mathcal{V}_f^h(\omega_i^L)$ for $j \in \mathcal{M}_i$ s.t

$$a(\tilde{\mathcal{T}}\phi_j, v_f) = -a(\phi_j, v_f), \quad \forall v_f \in \mathcal{V}_f^h(\omega_i^L),$$
$$a(\tilde{U}_{f,i}, v_f) = l(\Phi_i v_f), \quad \forall v_f \in \mathcal{V}_f^h(\omega_i^L).$$

## Coarse scale equation

Find $U_c \in \mathcal{V}_c$ s.t

$$a(U_c + \tilde{\mathcal{T}} U_c, v_c) = l(v_c) - (\tilde{U}_f, v_c), \quad \forall v_c \in \mathcal{V}_c.$$

# Decay in $V_f$

## Problem setting

- Let the computational domain be $\omega_i^L$ for $L = 1, 2, \ldots, N$ where $\omega_i^L \subseteq \Omega$.
- Let also $\Phi_i = \sum_{j \in \mathcal{M}_i} \phi_j$
- The problem reads: find $U \in \mathcal{V}^h(\omega_i^L)$

$$a(U, v) = -a(\Phi_i, v), \quad \forall v \in \mathcal{V}^h(\omega_i^L).$$

- The reference solution $U_{ref}$ is the solution computed on $\omega_i^N = \Omega$.

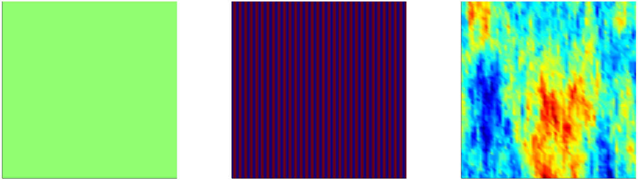


Figure: Permeabilities $\alpha$.

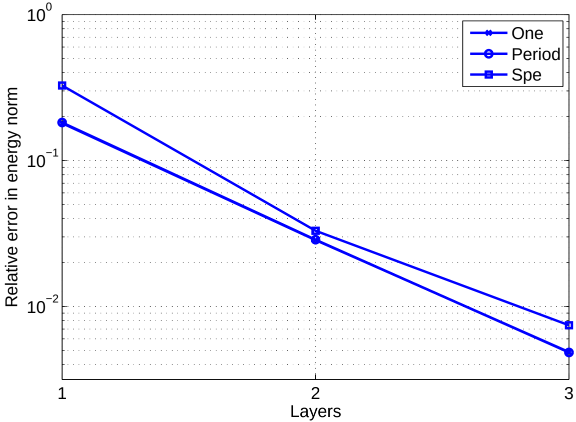



Figure: The error in relative error in broken energy norm with respect to the path size.

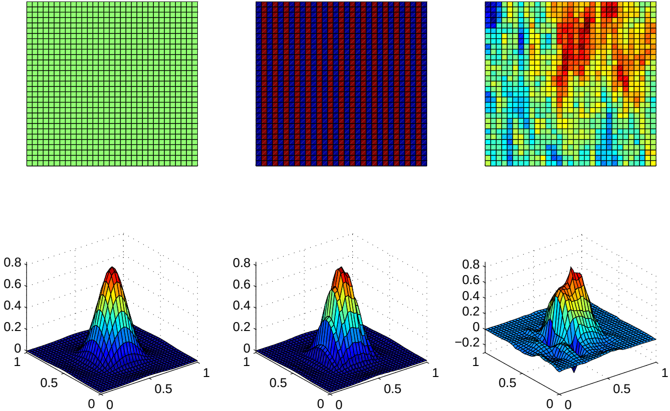
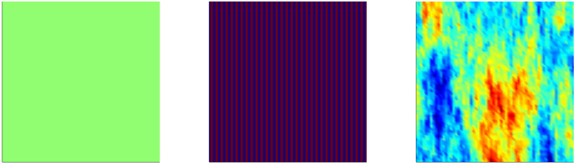



Figure: Example of $U + \Phi_i$ computed on 2 layer patches.

# Convergence

## Problem setting

- Consider the model problem (Poisson's equation)
- Keeping the refinement level constant and increasing the patch sizes $L = 1, \ldots, N$ for all local problems.
- The reference solution $U_{ref}$ is the DG solution computed on the fine scale.



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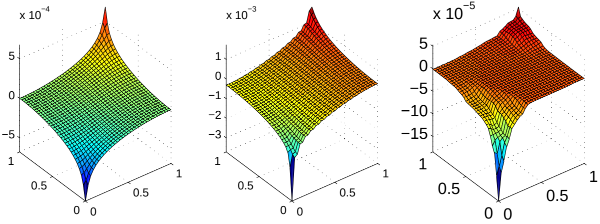


Figure: The reference solution to the model problem using the permeabilities *One*, *Period* and *SPE*

# Convergence

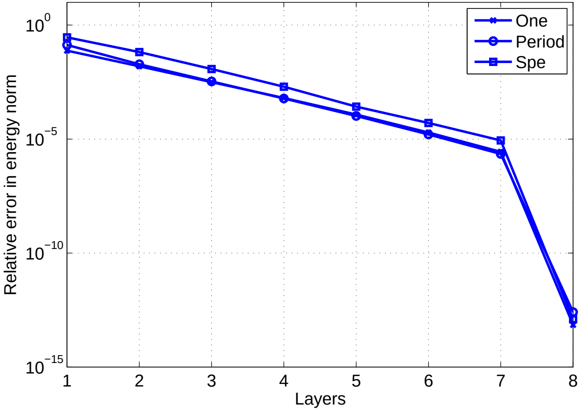


Figure: The relative error in broken energy norm with respect to the patch sizes.

# A posteriori error estimate

- Let $\mathscr{E} = u - U$ where $U = U_c + \tilde{\mathcal{T}} U_c + U_f$.
- $U_i = \sum_{j \in \mathcal{M}_i} U_{c,j}(\phi_j + \tilde{\mathcal{T}}\phi_j) + U_{f,i}$.
- $\lesssim \Leftrightarrow \leq c$ when the constant $c$ is independent of $H$, $h$ and $L$.

## Theorem (A posteriori error estimate)

*The error $\mathscr{E}$ satisfies the estimate*

$$\Big( \sum_{K \in \mathcal{K}} ||\sqrt{\alpha}\nabla \mathscr{E}||^2_{L^2(K)} \Big)^{1/2} \lesssim \sum_{K \in \mathcal{K}_c} \rho^2_{h,K} + \sum_{i \in \mathcal{N}} \rho^2_{L,\omega_i^L},$$

*where*

$$\rho^2_{L,\omega_i^L} = \sum_{e \in \Gamma^B(\omega_i^L) \setminus \Gamma^B} \rho^2_{L,\omega_i^L,e},$$

$$\rho_{L,\omega_i^L,e} = \frac{H_{\omega_i^L}}{\sqrt{h_e \alpha_0}} ||n \cdot \{\alpha \nabla U_i\}||_{L^2(e)} + \frac{H_{\omega_i^L}\sqrt{\sigma_e}}{h_e^{3/2}} ||[U_i]||_{L^2(e)},$$

and

$$\rho_{h,K}^2 = \sum_{K \in K_c} \frac{h}{\sqrt{\alpha_0}} ||f + \nabla \cdot \alpha \nabla U||_{L^2(K)}$$
$$+ \sum_{e \in \Gamma^I(K_c)} \left[ \sqrt{\frac{h_K}{\alpha_0}} ||n \cdot [\alpha \nabla U]||_{L^2(e)} + \sqrt{\frac{\sigma}{h_K}} ||[U]||_{L^2(e)} \right]$$
$$+ \sum_{e \in \Gamma^B(K_c) \setminus \Gamma^B} \frac{1}{2} \left[ \sqrt{\frac{h_K}{\alpha_0}} ||n \cdot [\alpha \nabla U]||_{L^2(e)} + \sqrt{\frac{\sigma}{h_K}} ||[U]||_{L^2(e)} \right].$$
$$+ \sum_{e \in \Gamma^B(K_c) \cap \Gamma^B} \sqrt{\frac{h_K}{\alpha_0}} ||n \cdot \alpha \nabla U||_{L^2(e)},$$

- $\rho_{L,\omega_i^L}^2$ measure the effect of the truncated patches.
- $\rho_{L,K}^2$ measure the effect of the refinement level.

# Adaptivity

- Consider the model problem
- Using the a posteriori error estimate to construct an adaptive algorithm.
- Start with one refinement and 2 layers patches everywhere.
- Refine 30% of the coarse elements and increase 30% of the patch sizes in each iteration.

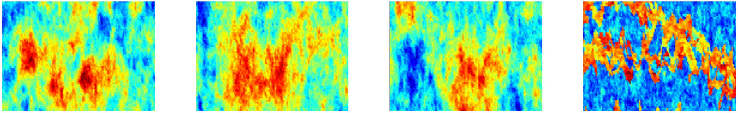

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Figure: Permeabilities $\alpha$ projection in log scale.

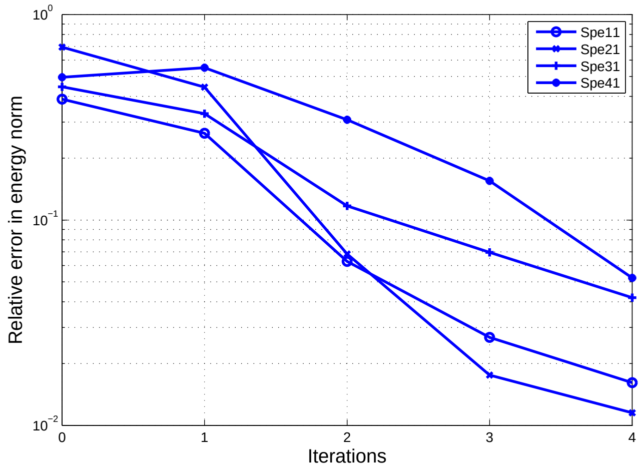
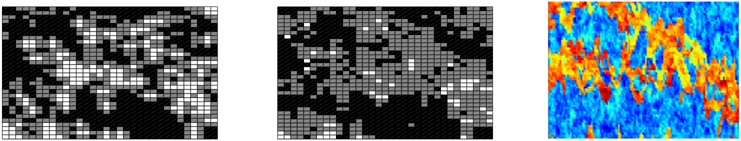



Figure: The relative error in broken energy norm with respect to number of iterations. Iteration 0 corresponds to the standard DG solution and iteration 1 the start values in the adaptive algorithm.

Figure (a) and (b) illustrates where the adaptive algorithm puts most effort

- Figure (a) corresponds to the refinements
- Figure (b) corresponds to the patch sizes.
- Figure (c) is the permeability $\alpha$.



(a) Refine $h_K$ (b) Layers, L (c) Layers, L

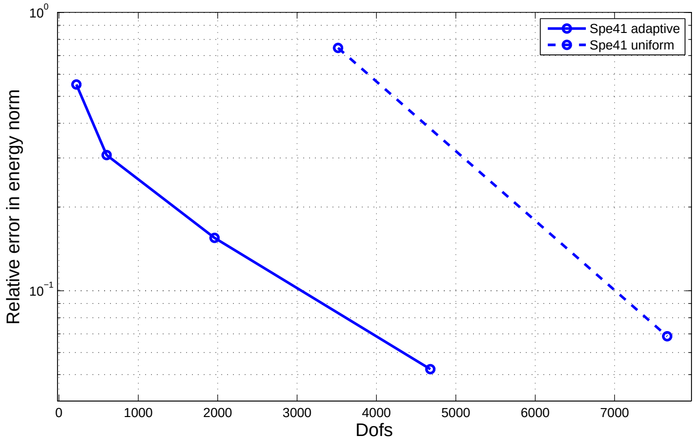



Figure: The relative error in broken energy norm with respect to the mean value of the degrees of freedom for the fine scale problems.

# Conclusions

## Advantage

- The fine scale problems are perfectly parallelizable.
- The exponential decay in the fine scale solution allows small patches.
- The error estimate and the adaptivity algorithm focus computational effort in critical areas.
- Very high aspect ratio in $\alpha$ can be solved.
- Possible to construct a conservative flux on the coarse scale.

# Future work

- Using a discontinuous Galerkin method with weighted average.
- Solve the local problems iteratively with a appropriate preconditioner. Condition number scale nicely with H,h but not with $\alpha_{max}/\alpha_{min}$.
- Convergence of the discontinuous Galerkin multiscale method.
- 3D implementation.

# Questions