UPPSALA
UNIVERSITET

A discontinuous Galerkin local orthogonal decomposition method for elliptic multiscale problems

Daniel Elfverson
daniel.elfverson@it.uu.se

Division of Scientific Computing
Uppsala University
Sweden

Model problem

Consider the convection-diffusion problem

$$\begin{aligned} -\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

where

- $0 < A_{min} \in \mathbb{R} \leq A(x) \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$,
- $f \in L^2(\Omega)$,
- $\mathbf{b} \in [W^1_\infty(\Omega)]^d$, and $\nabla \cdot \mathbf{b} = 0$.

Discontinuous Galerkin discretization

- $a_h(\cdot,\cdot)$: symmetric interior penalty (SIPG) and upwind.
- The energy-norm is defined by

$$|||\cdot|||_H^2 = ||A^{1/2}\nabla_H \cdot||_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}} \left(\frac{\sigma}{H} + \frac{|\mathbf{b}\cdot\nu|}{2}\right)\|[\cdot]\|_{L^2(e)}^2$$

- Let $\mathcal{V}_H$ be the space of discontinuous piecewise (bi)linear polynomials.

(One scale) DG method

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

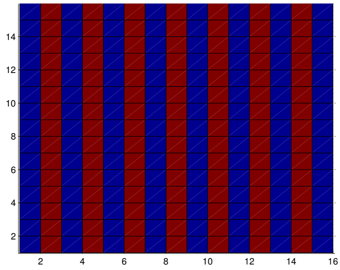


Figure : The coefficient A in the model problem.

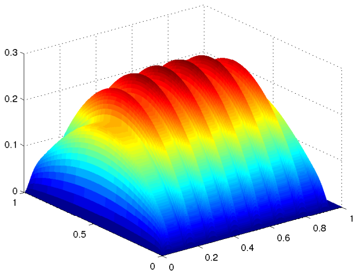


Figure : Reference solution.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

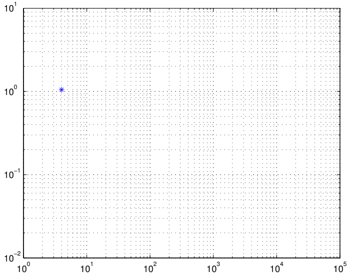


Figure : Energy norm with respect to the degrees of freedom.

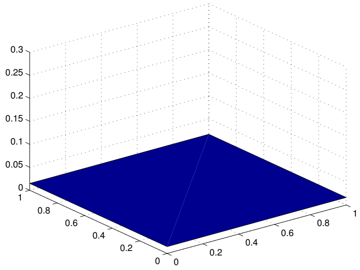


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

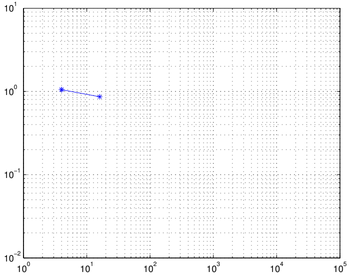


Figure : Energy norm with respect to the degrees of freedom.

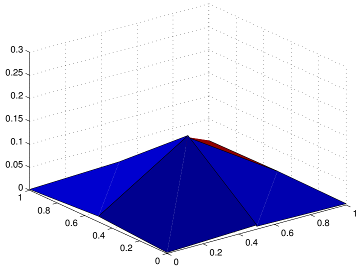


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

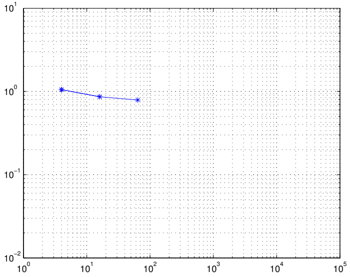


Figure : Energy norm with respect to the degrees of freedom.

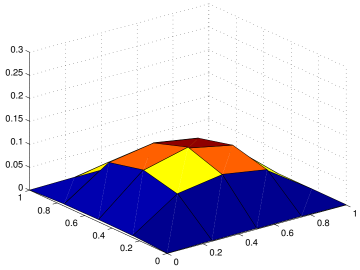


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

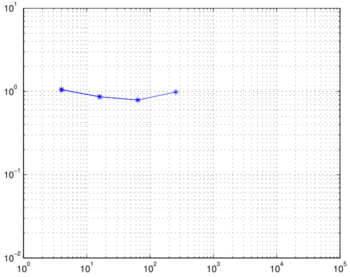


Figure : Energy norm with respect to the degrees of freedom.

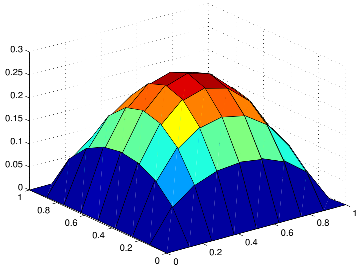


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

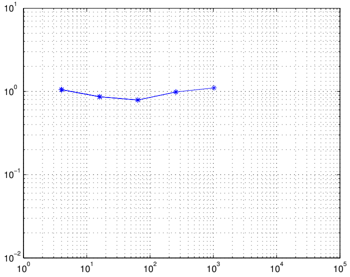


Figure : Energy norm with respect to the degrees of freedom.

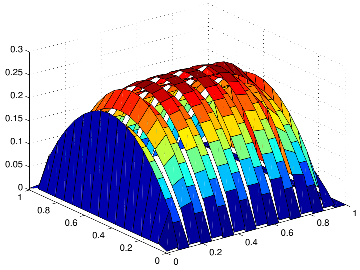


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

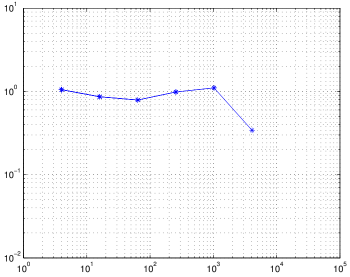


Figure : Energy norm with respect to the degrees of freedom.

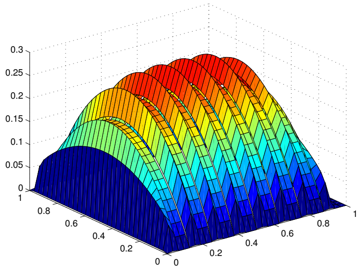


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

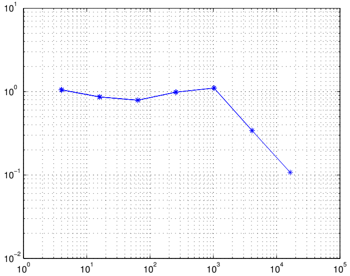


Figure : Energy norm with respect to the degrees of freedom.

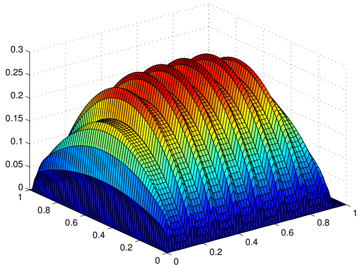


Figure : Solution obtained using the discontinuous Galerkin method.

(One scale) DG method ($\mathbf{b} = 0$)

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

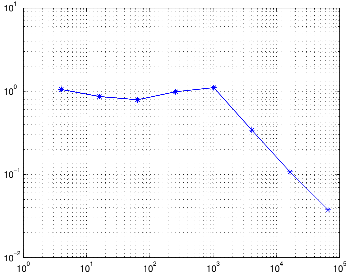


Figure : Energy norm with respect to the degrees of freedom.

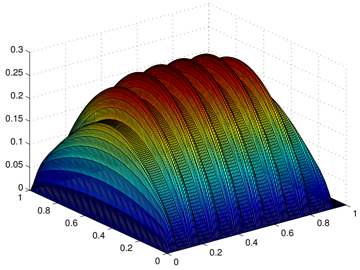


Figure : Solution obtained using the discontinuous Galerkin method.

Multiscale split

- Consider $\mathcal{V}_H$ and $\mathcal{V}_h$, such that $\mathcal{V}_H \subset \mathcal{V}_h$.
- Let Π_H be the L^2-projection onto $\mathcal{V}_H$.
- Define $\mathcal{V}^f(\omega) = \{v \in \mathcal{V}_h(\omega) : \Pi_H v = 0\}$.
- We have a L^2-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}^f$.

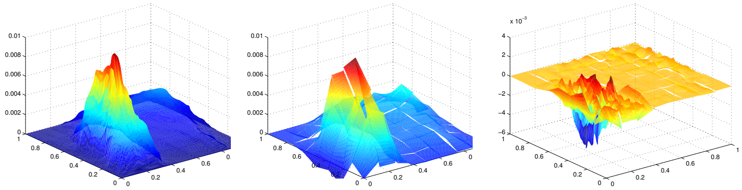


Figure : $u_h = u_H + u^f$

Corrected basis functions

- For each $\lambda_{T,j} \in \mathcal{V}_H$ we compute a corrector, find $\phi^L_{T,j} \in \mathcal{V}^f(\omega^L_T)$ such that

$$a_h(\phi^L_{T,j}, v_f) = a_h(\lambda_{T,j}, v_f), \quad \text{for all } v_f \in \mathcal{V}^f(\omega^L_T).$$

  where L indicates the size of the patch.
- Corrected space: $\mathcal{V}_H^{ms} = \text{span}\{\lambda_{T,j} - \phi^L_{T,j}\}$.
- We have a $a(\cdot,\cdot)$-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H^{ms} \oplus \mathcal{V}^f$.

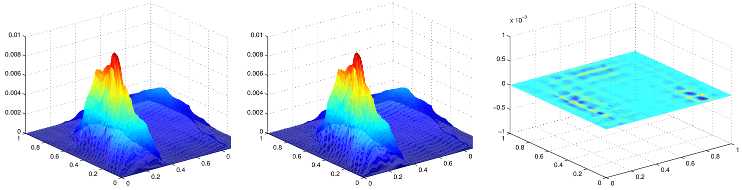
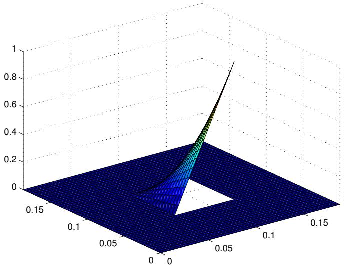
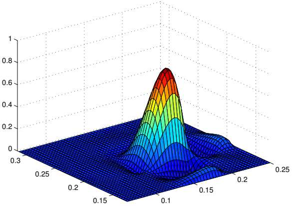
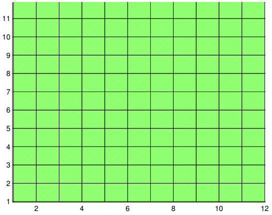


Figure : $u_h = u_H^{ms} + u^f$

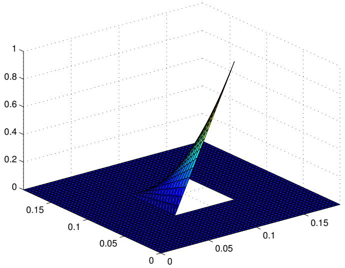
Examples of corrected basis functions

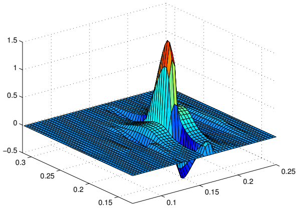


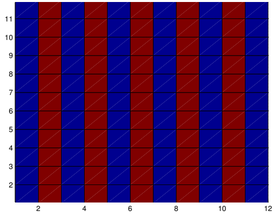




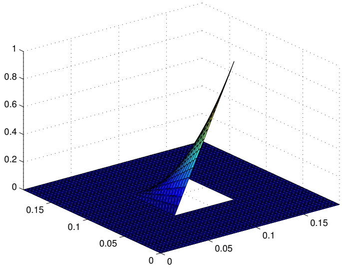
Examples of corrected basis functions

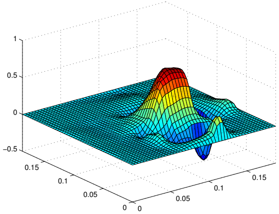


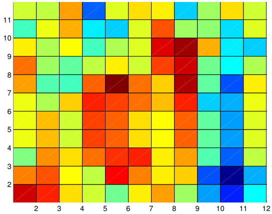




Examples of corrected basis functions

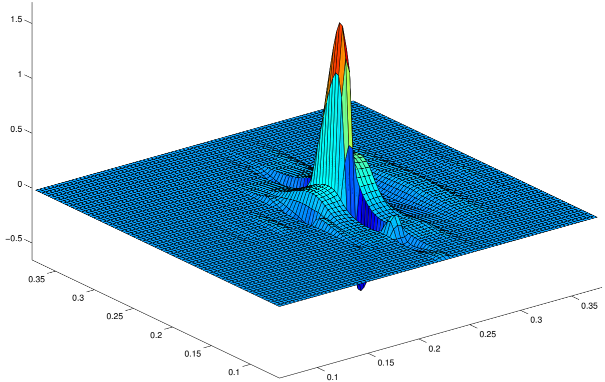






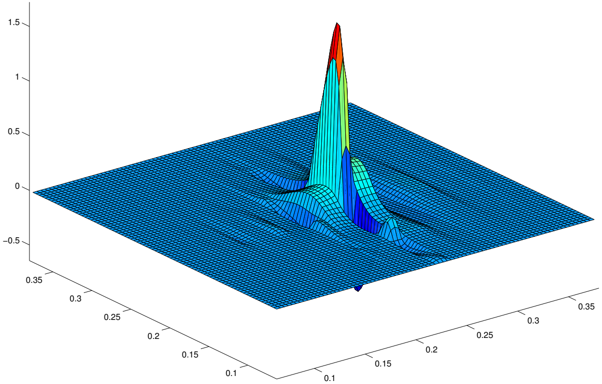
Example of corrected basis function

- With $\mathbf{b} = [0, 0]'$.



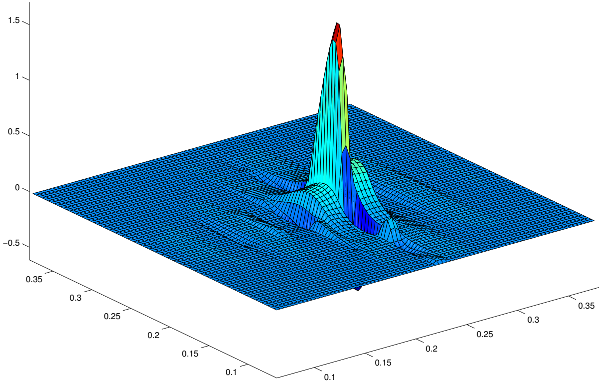
Example of corrected basis function

- With $\mathbf{b} = -[1, 0]'$.



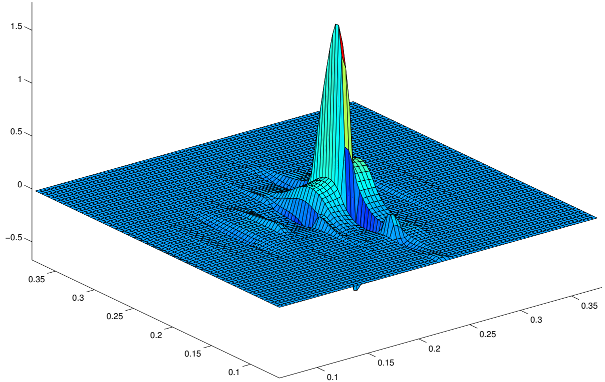
Example of corrected basis function

- With $\mathbf{b} = -[2, 0]'$.



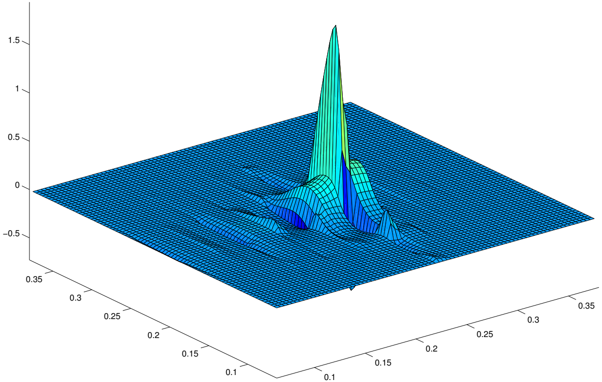
Example of corrected basis function

- With $\mathbf{b} = -[4, 0]'$.



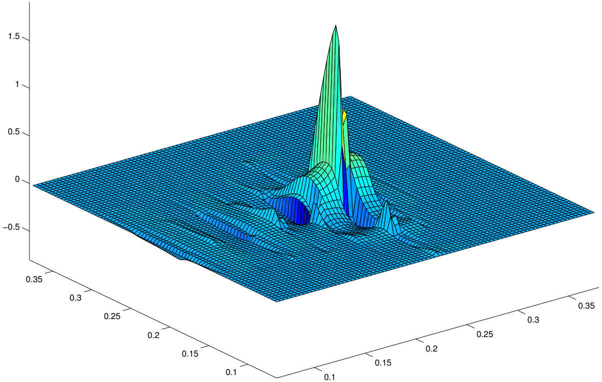
Example of corrected basis function

- With $\mathbf{b} = -[8, 0]'$.



Example of corrected basis function

- With $\mathbf{b} = -[16, 0]'$.



Discontinuous Galerkin multiscale method

Consider the problem: find $u_H^{ms,L} \in \mathcal{V}_L^{ms} = \text{span}\{\lambda_{T,j} - \phi_{T,j}^L\}$ such that

$$a_h(u_H^{ms,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H^{ms,L}.$$

- $\text{dim}\mathcal{V}_H^{ms,L} = \text{dim}\mathcal{V}_H$
- The basis function are solved independently of each other.
- Method can take advantage of periodicity.

A priori error bound

Under the assumption $\mathcal{O}(\|H\mathbf{b}\|_{L^\infty(\Omega)}/A_{min}) = 1$ it holds:

Lemma (Decay of corrected basisfunctions)

For $\phi_{T,j} \in \mathcal{V}^f(\omega_i^L)$, there exist a, $0 < \gamma < 1$, such that

$$|||\phi_{T,j} - \phi_{T,j}^L||| \lesssim \gamma^L |||\lambda_j - \phi_{T,j}|||.$$

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$, there exist a, $0 < \gamma < 1$, such that

$$|||u - u_H^{ms,L}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L ||f||_{L^2}.$$

Choosing $L = \lceil C \log(H^{-1}) \rceil$ both terms behave in the same manor with an appropriate C.

Numerical verification of the convergence

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

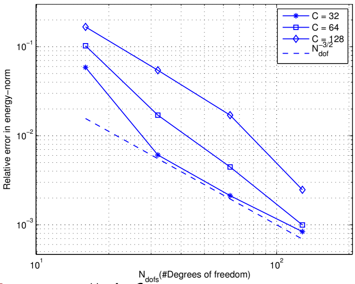


Figure : $\#dofs$ vs $|||u_h - u_{H,L}^{ms}|||/|||u_h|||$

- Let $A = 1$ and $\mathbf{b} = C[1,0]'$ for $C = 32, 54, 128$.
- Choose $L = \lceil 2\log(\frac{1}{H})\rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{2, 3, 4, 5\}$.
- Reference mesh is 2^{-7}.

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $\mathbf{b} = [1, 0]'$.

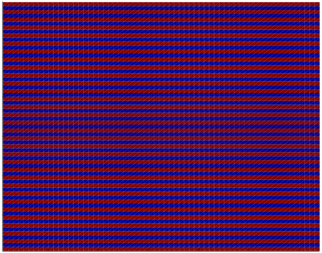


Figure : Diffusion coefficient A, $A_{max}/A_{min} = 100$ and $A_{min} = 0.01$.

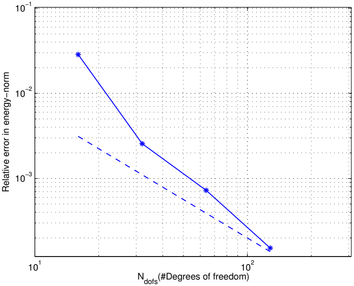


Figure : $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $\mathbf{b} = [512, 0]'$.

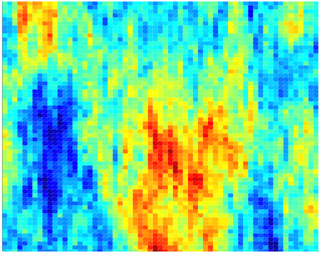


Figure : Diffusion coefficient A with $A_{max}/A_{min} \sim 10^5$ and $A_{min} = 0.05$.

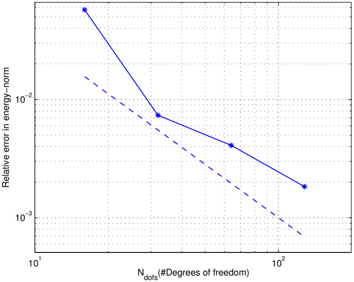


Figure : $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

Adaptivity and a posteriori error bound ($\mathbf{b} = 0$)

Theorem (A posteriori error bound)

Let $u_H^{ms,L}$ be the multiscale solution, then

$$|||u - u_H^{ms,L}||| \lesssim \left(\sum_{T \in \mathcal{T}_H} \rho_{h,T}^2(u_H^{ms,L}) \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_H} \rho_{L,\omega_T^L}^2(u_H^{ms,L}) \right)^{1/2}.$$

- $\rho_{L,\omega_i^L}^2$ measures the effect of the truncated patches.
- $\rho_{h,T}^2$ measures the effect of the refinement level.

- We consider the permeabilities

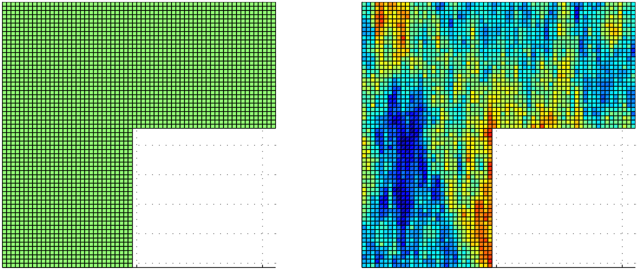


Figure : Permeabilities *One* left and *SPE* right.

- Using a refinement level of 30% we have.

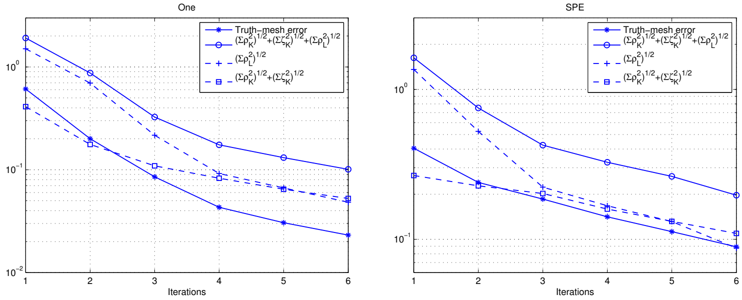


Figure : Convergence plot for *One* left and *SPE* right.

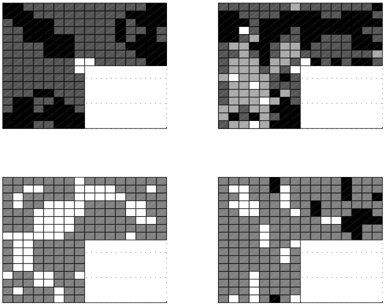


Figure : One (left) and SPE (right). The level of refinement (upper) and size of the patches (lower).

Perspective towards Two-Phase flow

Buckley-Leverett system

$$-\nabla \cdot (K\lambda(S)\nabla p) = q \text{ and } \partial_t S + \nabla \cdot (f(s)\mathbf{v}) = q_w$$

is solved using IMPES. Here

- K is the hydraulic conductivity,
- $\lambda(S)$ is the total mobility (essentially macroscopic),
- and $\mathbf{v} = -K\lambda(S)\nabla p$ is obtained from the pressure equation.

- Coarse mesh $H = 2^{-5}$ and fine mesh $h = 2^{-8}$.
- Boundary condition $p = 1$, on left boundary $p = 0$ on right boundary, and $K\lambda(S)\nabla p = 0$ otherwise.
- Prepossessing step: compute the basis corrected basis using $\lambda(S) = 1$

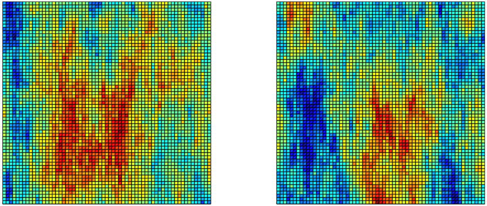


Figure : K_1 ($A_{max}/A_{min} \approx 5 \cdot 10^5$) left and K_2 ($A_{max}/A_{min} \approx 4 \cdot 10^5$) right on a mesh with size 2^{-6}.

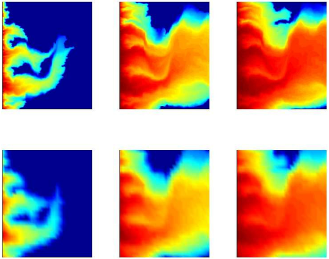


Figure : Saturation profile K_1 for T_1, T_2, and T_3.

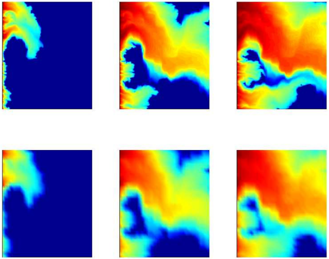


Figure : Saturation profile K_2 for T_1, T_2, and T_3.

Data	$\|e(T_1)\|_{L^2(\Omega)}$	$\|e(T_2)\|_{L^2(\Omega)}$	$\|e(T_3)\|_{L^2(\Omega)}$
1	0.088	0.073	0.070
2	0.058	0.087	0.079

Table : Error in relative L^2-norm, $e(T) = S(T) - S^{\mathrm{ref}}(T)$.

D. Elfverson, G. H. Georgoulis, and A. Målqvist
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *Multiscale Model. Simul.*.

D. Elfverson, G. H. Georgoulis, A. Målqvist and D. Peterseim
Convergence of discontinuous Galerkin multiscale methods. *SIAM J. Numer. Anal.*.

D. Elfverson
A discontinuous Galerkin multiscale method for convection-diffusion problems. *Submitted*.

D. Elfverson, V. Ginting, P. Henning
On Multiscale Methods in Petrov-Galerkin formulation. *arXiv:1405.5758, submitted*.