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Discontinuous Galerkin multiscale methods for elliptic problems

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Outline

Elfverson, Georgoulis, and Målqvist
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *Submitted.*

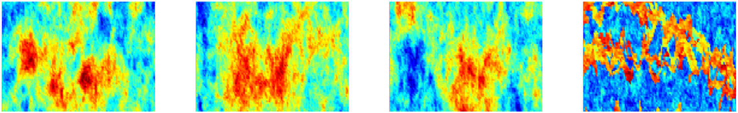
Elfverson, Georgoulis, Målqvist and Peterseim
Localization of discontinuous Galerkin multiscale methods. *In preparation.*

Model problem

Given a polygonal domain $\Omega \subset \mathbb{R}^d$: find $u \in H^1(\Omega)$ such that

$$\begin{aligned} -\nabla \cdot \alpha \nabla u &= f \text{ in } \Omega, \\ n \cdot \alpha \nabla u &= 0 \text{ on } \partial\Omega, \end{aligned}$$

for $0 < \alpha_{min} \leq \alpha(x) \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $\int_\Omega f \, dx = 0$.



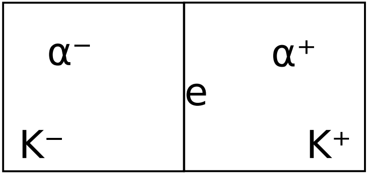
(a) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (b) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (c) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (d) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^6$

Figure: Permeabilities α projected in log scale and taken from the Society of Petroleum Engineer `http://www.spe.org/`.

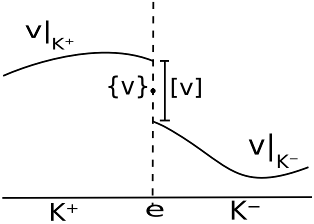
Discontinuous Galerkin discretization

- Consider the partition $\mathcal{K} = \{K\}$ and let Γ be the union of all edges.
- Let also $\mathcal{V}_h$ be the space of all discontinuous piecewise (bi)linear polynomials.
- Define the weighted average and jump on face e as:

$$\{v\}_w = \frac{\alpha^+ v^-}{\alpha^+ + \alpha^-} + \frac{\alpha^- v^+}{\alpha^+ + \alpha^-} \quad \text{and} \quad [v] = v^+ - v^-.$$



(a) Here $\mathcal{K} = \{K^+, K^-\}$ and $\Gamma^I = \{e\}$



(b) Example of $\{v\}$ and $[v]$

Let

$$
\begin{aligned}
a(v,z) = \sum_{K\in\mathcal{K}} (\alpha\nabla v, \nabla z)_{L^2(K)} - \sum_{e\in\Gamma^I} \Big((\mathbf{n}\cdot\{\alpha\nabla v\}_w, [z])_{L^2(e)} \\
+ (\mathbf{n}\cdot\{\alpha\nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e\gamma_e}{h_e}([v],[z])_{L^2(e)}\Big), \\
F(v) = (f,v)_{L^2(\Omega)}.
\end{aligned}
$$

where

$$
|||v|||^2 = \sum_{K\in\mathcal{K}} \|\sqrt{\alpha}\nabla v\|^2_{L^2(K)} + \sum_{e\in\Gamma} \frac{\sigma_e\gamma_e}{h} \|[v]\|^2_{L^2(e)}
$$

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$
a(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.
$$

Example

Let $\alpha = \alpha(x/\epsilon)$. We have the known result for periodic coefficients

$$|||u - u_H||| \leq C\frac{H}{\epsilon}||f||_{L^2(\Omega)}.$$

- Need $H < \epsilon$ for reliable results, computational prohibitive to solve on a single mesh.

Note: From now on we only consider $0 < \alpha_{min} \leq \alpha(x) \in L^{\infty}(\Omega)$ without any assumtions on scale or periodicity.

Objective

- Eliminate the ϵ-dependence via a multiscale method i.e.,

$$|||u - u_H^{ms}||| \leq C(f)H.$$

- Construct an adaptive algorithm to focus computational effort in critical areas.

Some known methods

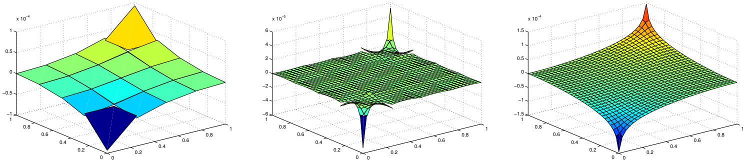
- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98.
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09.
- Multiscale FEM: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06.
- Residual free bubbles: Brezzi et al. 98.
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zang 04, Ohlberger 05.
- Equation free: Kevrekidis et al. 05.
- Metric based upscaling: Owhadi-Zang et al. 06.
- GFEM: Babuška-Lipton 2011.

Remarks

- Local approximations (in parallel) on a fine scale are used to modify a coarse scale space or equation.

Variational multiscale framework

- Consider a coarse mesh $\mathcal{K}_H \subset \mathcal{K}_h$.
- Let $\mathcal{V}_H = span\{\phi_i\} = \Pi_H \mathcal{V}_h$ and $\mathcal{V}_f = \{v \in \mathcal{V}_h : \; \Pi_H v = 0\}$, where $\Pi_H : \mathcal{V}_h \to \mathcal{V}_H$ is the L^2 projection onto the coarse mesh.
- The problem is split into one coarse and fine scale contribution $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}_f$.



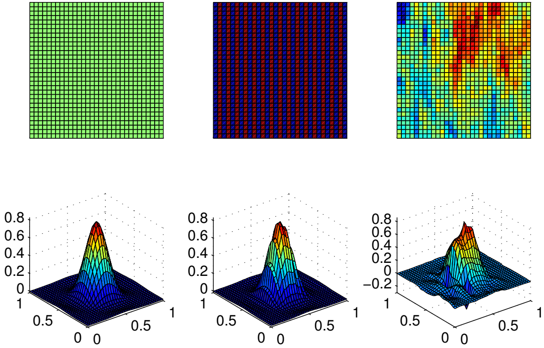
(c) u_H (d) u_h^f (e) $u_h = u_H + u_h^f$

View solution as span of modified basis functions

- Define the map $\mathcal{T} : \mathcal{V}_H \to \mathcal{V}_f$ as

$$a(\mathcal{T} v_H, v_f) = -a(v_H, v_f), \quad \forall v_H \in \mathcal{V}_H, v_f \in \mathcal{V}_f.$$

- We let $\mathcal{V}^{ms} = \mathcal{V}_H + \mathcal{T}\mathcal{V}_H = \text{span}\{\phi_i + \mathcal{T}\phi_i\}$.
- $\phi_i + \mathcal{T}\phi_i$ can be viewed as a coarse modified basis function.
- From the multiscale map we have, $\mathcal{V}_h = \mathcal{V}_{ms} \oplus_a \mathcal{V}_f$.



Localization of $\mathcal{T}\phi_i$

- For each i we have, $a(\mathcal{T}_i^L\phi_i, v) = -a(\phi_i, v)$ for all $v \in \mathcal{V}_f(\omega_i^L)$, solved on local Dirichlet or Neumann patches.
- Define the localized multiscale space by, $\mathcal{V}_L^{ms} := \text{span}\{\phi_i + \mathcal{T}_i^L\phi_i\}$.

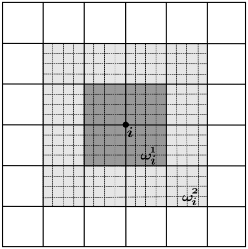


Figure: Example of a one layer patch ω_i^1 and a two layer patch ω_i^2

A priori results

Consider the problem: find $u^{ms}_{H,L} \in \mathcal{V}^{ms}_L = \text{span}\{\phi_i + \mathcal{T}_i^L \phi_i\}$ such that

$$a(u^{ms}_{H,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}^{ms}_L.$$

Lemma (Decay of modifed basisfunction)

For $\mathcal{T}_i^L \phi_i \in \mathcal{V}_f(\omega_i^L)$, there exist a, $0 < \gamma < 1$, such that

$$|||\mathcal{T}\phi_i - \mathcal{T}_i^L \phi_i||| \lesssim \gamma^L |||\phi_i + \mathcal{T}\phi_i|||_{\omega_i^L}.$$

Theorem

For $u^{ms}_{H,L} \in \mathcal{V}^{ms}_L$, there exist a, $0 < \gamma < 1$, such that

$$|||u - u^{ms}_{H,L}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L ||f||_{L^2}.$$

Note: Theorem holds without any assumptions on scales or regularity!

Numerical verification

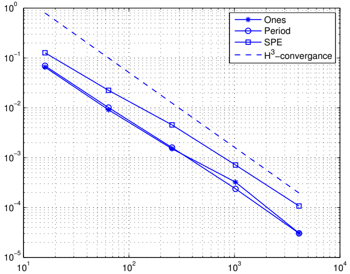


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Choose $L = \lceil 2\log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5\}$.
- Reference mesh is 2^{-7}.

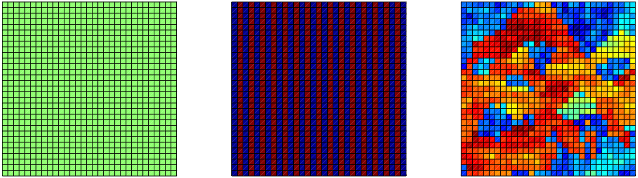


Figure: Permeabilies are piecewise constant on a mesh with size 2^{-5}, with ratio $\alpha_{max}/\alpha_{min} = \{1, 10, 7 \cdot 10^6\}$

Adaptivity

- Construct an adaptive algorithm to automatically tune the fine mesh size and the patch sizes.
- We now consider a non-symmetric coarse scale problem, using local Neumann problems for the modified basis functions, and using a right hand side correction.

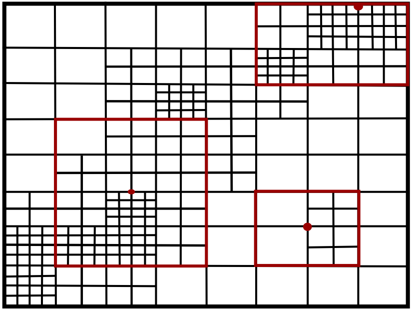


Figure: Example of an adapted mesh with varying patch sizes.

- Let $\mathcal{N}$ be the set of all coarse nodes and $\mathcal{M}_i$ be the set of all j such that $\phi_j(x_i) = 1$.
- Let $\tilde{\mathcal{V}}^{ms} = \text{span}\{\phi_j + \mathcal{T}_i^{L(i)}\phi_j\}$, with varing patch sizes.
- Let $U_h^f = \sum_{i\in\mathcal{N}} U_{h,i}^f$ be a right hand side correction obtaind by solving: find $U_{h,i}^f \in \mathcal{V}_f(\omega_i^{L(i)})$ such that

$$a(U_{h,i}^f, v) = F(v), \quad \text{for all } v \in \mathcal{V}_f(\omega_i^{L(i)}).$$

Coarse equation (with right hand side correction)

We consider: find $U^{ms} \in \tilde{\mathcal{V}}^{ms}$ such that

$$a(U^{ms}, v) = F(v) - a(U_h^f, v), \quad \text{for all } v \in \mathcal{V}_H.$$

where the multiscale solution is $U = U^{ms} + U_h^f$.

Theorem (A posteriori error estimate for ADG-MS)

Let $U_i := \sum_{j\in\mathcal{M}_i} U_j^{ms} + U_{f,i}$. *Then,*

$$|||\nabla(u-U)|||^2 \lesssim \sum_{K_H\in\mathcal{K}_H} \rho^2_{h,K_H} + \sum_{i\in\mathcal{N}} \rho^2_{L,\omega_i^L},$$

where

$$\rho^2_{h,K_H} = \sum_{K\in\mathcal{K}(K_H)} \frac{h_K}{\sqrt{\alpha_0}} ||f+\nabla\cdot\alpha\nabla U||_{L^2(K)} + \sqrt{\frac{h_K}{\alpha_0}}\Big(||(1-w_{K(e)})n\cdot[\alpha\nabla U]||_{L^2(\partial K)} + ||\frac{\sigma_e\gamma_e}{h_e}[U]||_{L^2(\partial K\setminus\Gamma^B)}\Big),$$

$$\rho^2_{L,\omega_i^L} = \sum_{e\in\Gamma^B(\omega_i^L)\setminus\Gamma^B} \left(\frac{H_{\omega_i^L}}{\sqrt{h_K\alpha_0}}\Big(||n\cdot\{\alpha\nabla U_i\}_w||_{L^2(e)} + \frac{\sigma_e\gamma_e}{h_e}||[U_i]||_{L^2(e)}\Big)\right),$$

- ρ^2_{L,ω_i^L} measures the effect of the truncated patches.
- $\rho^2_{h,K}$ measures the effect of the refinement level.

Numerical experiment

- Refine 30% of the coarse elements and increase 30% of the patch sizes in each iteration.
- Coarse mesh is 32×32 elements and reference grid is 256×256 elements.
- The right hand side is -1 in the lower left corner and 1 in the upper right.

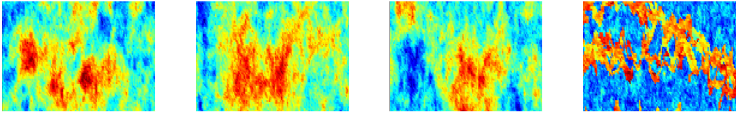


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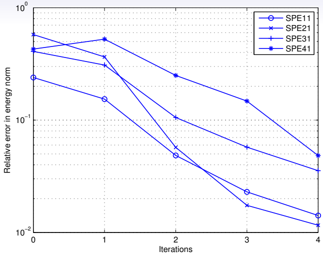
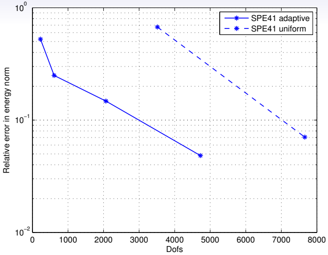
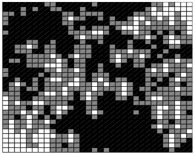
Figure: Permeabilities α projection in log scale.



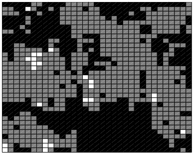
(a) The relative error in broken energy norm with respect to number of iterations. Iteration 0 corresponds to the standard DG solution and iteration 1 the start values in the adaptive algorithm.



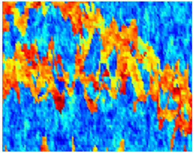
(b) The relative error in broken energy norm with respect to the mean value of the degrees of freedom for the fine scale problems.



(c) Refinement level, h_K



(d) Layers, L



(e) Permeability, α

Conclusions:

- The exponential decay in the modifed basis function allows small patches which are perfectly parallelizable.
- The error estimate and the adaptivity algorithm focus computational effort in critical areas.
- Get optimal convergance for the (crude) SPE Benchmark problem.
- DG: Flexibility in fine scale approximation spaces, boundary conditions and good conservation properties of the state variable

Futurework

- Using DG on the coarse scale but CG on the fine scale to save computational work.
- Construct an adaptive algorithem that increases the patch sizes only in the direction where the error is large.