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Discontinuous Galerkin multiscale methods for second order elliptic problems

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Outline

Model problem

Consider the elliptic problem

$$-\nabla \cdot A\nabla u + \Big(\mathbf{b} \cdot \nabla u + cu\Big) = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

where $0 < A_{min} \in \mathbb{R} \leq A(x) \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$, $\mathbf{b} \in [W^1_\infty(\Omega)]^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, with the standard assumption

$$c_o^2 = c - \frac{1}{2}\nabla \cdot \mathbf{b} \geq c_0 \in \mathbb{R} > 0.$$

Discontinuous Galerkin discretization

- Split Ω into a elements $\mathcal{T} = \{T\}$, and let $\mathcal{E} = \{e\}$ be the set of all edges in $\mathcal{T}$.

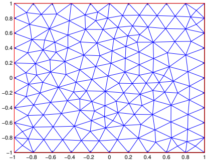

Figure: Example of a mesh on a unit square.

- Let $\mathcal{V}_H$ be the space of all discontinuous piecewise (bi)linear polynomials.

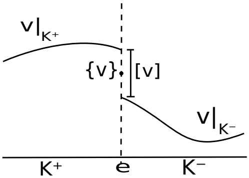

Figure: Example of $\{v\}$ and $[v]$

The bilinear form is defined by:

$$a_H(u,v) := a_H^{\mathrm{d}}(u,v) + a_H^{\mathrm{c\text{-}r}}(u,v).$$

where

$$a_H^{\mathrm{d}}(u,v) := (A\nabla_H u, \nabla_H v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_H} \Big(\frac{\sigma_e}{h_e}([u],[v])_{L^2(e)} - (\{\nu_e \cdot A\nabla u\},[v])_{L^2(e)} - (\{\nu_e \cdot A\nabla v\},[u]_{L^2(e)}) \Big),$$

where σ_e is a constant and

$$a_H^{\mathrm{c\text{-}r}}(u,v) := (\mathbf{b}\cdot\nabla_H u + cu, v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_H} (b_e[u],[v])_{L^2(e)} - \sum_{e\in\mathcal{E}_H(\Omega)} (\nu_e\cdot\mathbf{b}\{u\},[v])_{L^2(e)} - \sum_{e\in\mathcal{E}_H(\Gamma)} \frac{1}{2}((\nu_e\cdot\mathbf{b})u, v)_{L^2(e)},$$

where $b_e = |\nu_e \cdot \mathbf{b}|/2$.

- $a_H^{\mathrm{d}}(\cdot,\cdot)$ approximates the diffusion a interior penalty method.
- $a_H^{\mathrm{c\text{-}r}}(\cdot,\cdot)$ approximates the convection-reaction using upwind.

- The energy-norm is defined by

$$|||v|||_H^2 = ||A^{1/2}\nabla_H v||_{L^2(\Omega)}^2 + \|c_o v\|_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}}\left(\frac{\sigma}{H} + \frac{|\mathbf{b}\cdot\nu|}{2}\right)\|[v]\|_{L^2(e)}^2$$

- Let $\mathcal{V}_H$ be the space of discontinuous piecewise (bi)linear polynomials.

(One scale) DG method

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

(One scale) DG method for a Poisson's equation with variable coefficients

Find $u_H \in \mathcal{V}_H$ such that

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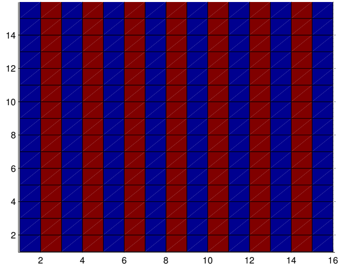

Figure: The coefficient A in the model problem.

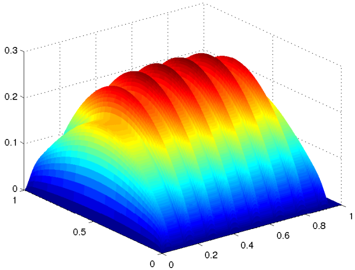

Figure: Reference solution.

(One scale) DG method for a Poisson's equation with variable coefficients

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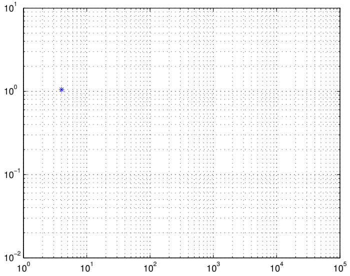

Figure: Energy norm with respect to the degrees of freedom.

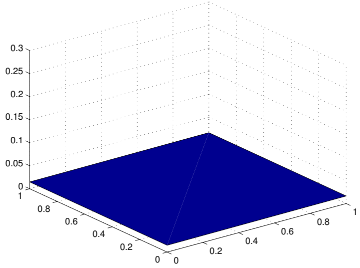

Figure: Solution obtained using the discontinuous Galerkin method.

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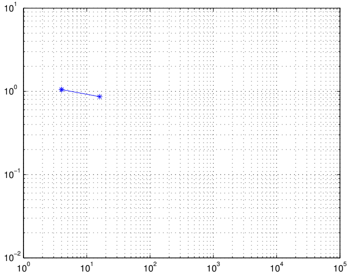

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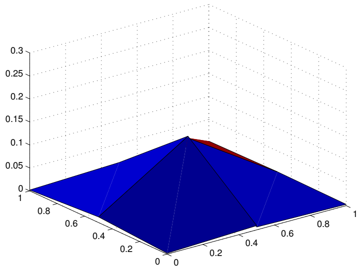

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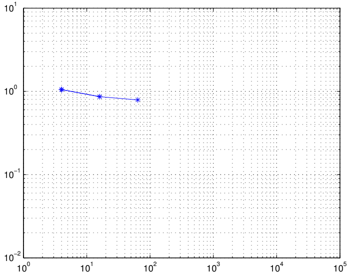

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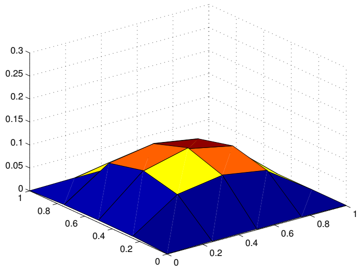

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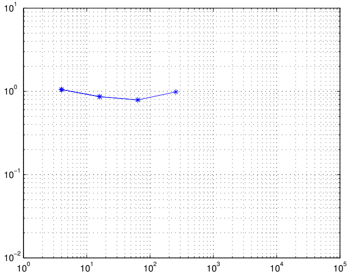

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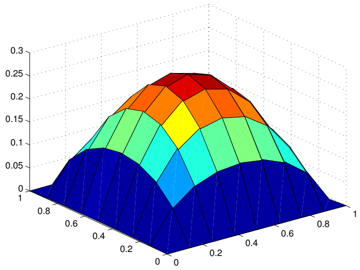

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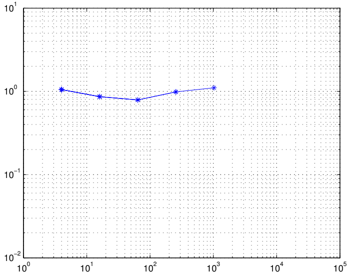

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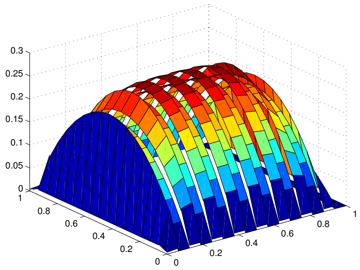

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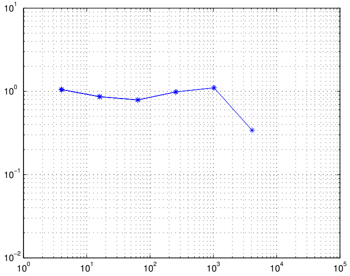

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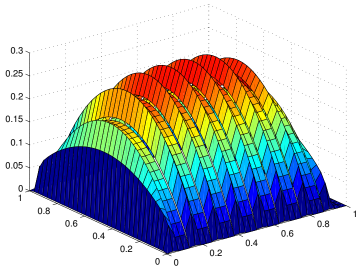

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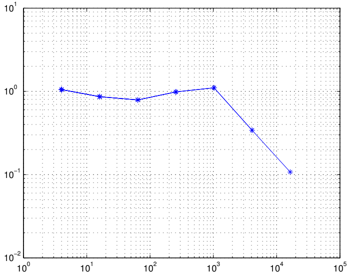

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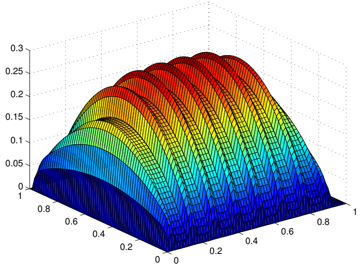

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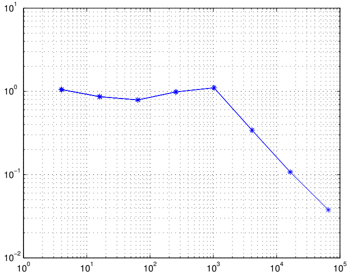

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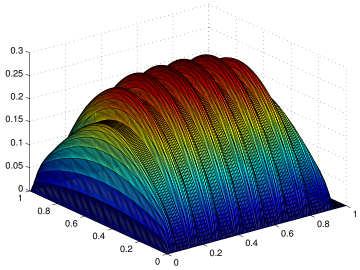


Figure: Solution obtained using the discontinuous Galerkin method.

Objective with the multiscale method

- Eliminate the dependency of A via a multiscale method i.e.,

$$|||u - u_H^{ms,L}||| \leq C_f H,$$

  where H does not resolve the variation in A
- Construct an adaptive algorithm to focus computational effort to critical areas (for Poisson's equation with variable coefficients).

Some known methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98.
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09.
- MsFEM: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06.
- Residual free bubbles: Brezzi et al. 98.
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zang 04, Ohlberger 05.
- Equation free: Kevrekidis et al. 05.
- GFEM: Babuska-Lipton et al. 11.
- Metric based upscaling: Owhadi-Zang et al. 06.
- Generalized MsFEM: Efendiev et al. 13.
- ...

Remarks

- Local approximations (in parallel) on a fine scale are used to modify a coarse scale space or equation.

Multiscale split

- Consider a coarse $\mathcal{V}_H$ and a fine space $\mathcal{V}_h$, such that $\mathcal{V}_H \subset \mathcal{V}_h$.
- Let Π_H be the L^2-projection onto $\mathcal{V}_H$. This will be used as the split between the coarse and fine scale.
- Define $\mathcal{V}^f(\omega) = \{v \in \mathcal{V}_h(\omega) : \Pi_H v = 0\}$.
- We have a L^2-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}^f$.

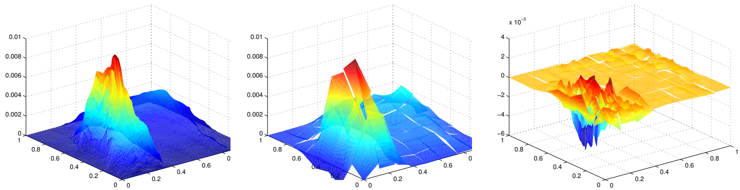


Figure: $u_h = u_H + u^f$

Corrected basis functions

- For each basis function $\lambda_{T,j} \in \mathcal{V}_H$ we calculate a corrector, find $\phi^L_{T,j} \in \mathcal{V}^f(\omega^L_T)$ such that

$$a_h(\phi^L_{T,j}, v_f) = a_h(\lambda_{T,j}, v_f), \quad \text{for all } v_f \in \mathcal{V}^f(\omega^L_T).$$

  where $\text{supp}(\lambda_{T,j}) = T$ and L indicates the size of the patch.
- Let the new corrected space be defined by $\mathcal{V}^{ms}_H = \text{span}\{\lambda_{T,j} - \phi^L_{T,j}\}$.
- We have an $u_h = u^{ms}_H + u^f$ where $u_h \in \mathcal{V}_h$, $u^{ms}_H \in \mathcal{V}^{ms}_H$, and $u^f \in \mathcal{V}^f$.

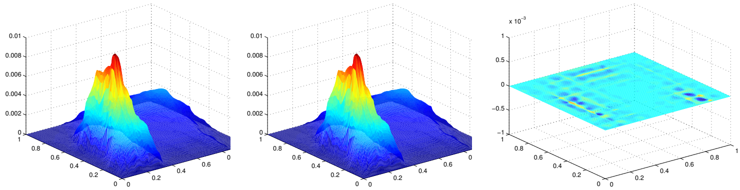
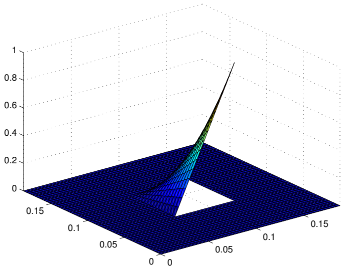
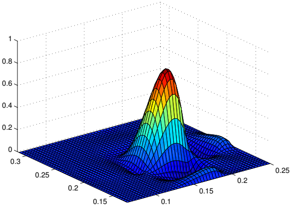
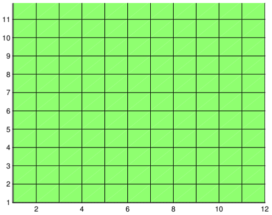


Figure: $u_h = u^{ms}_H + u^f$

Examples of corrected basis functions







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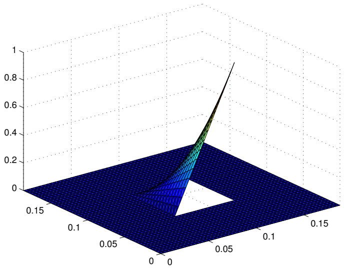

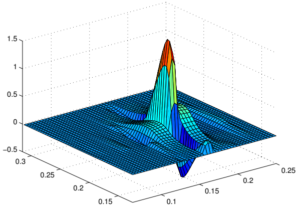

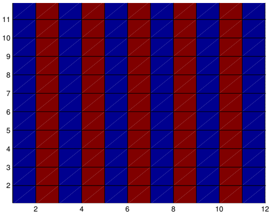

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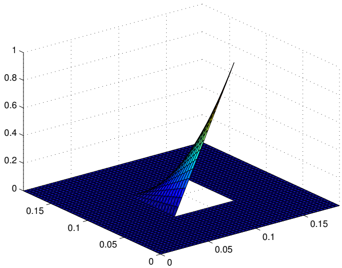
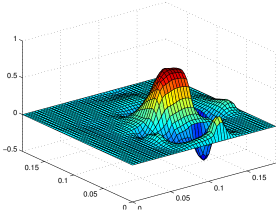
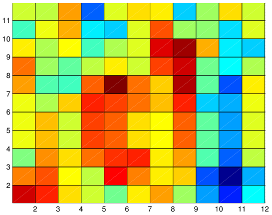
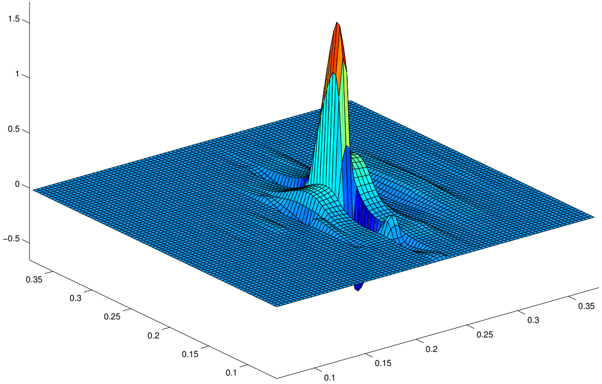






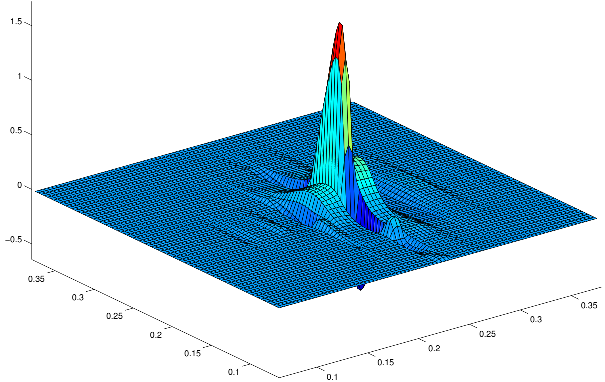
Example of corrected basis function

- With $\mathbf{b} = [0, 0]'$.



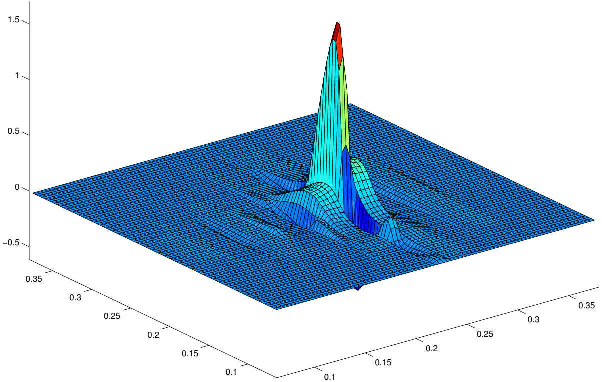
Example of corrected basis function

- With $\mathbf{b} = -[1, 0]'$.



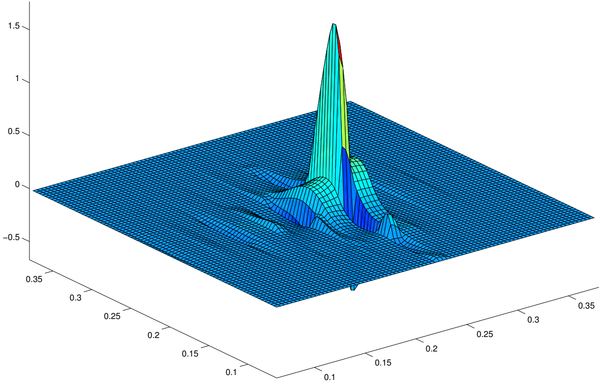
Example of corrected basis function

- With $\mathbf{b} = -[2, 0]'$.



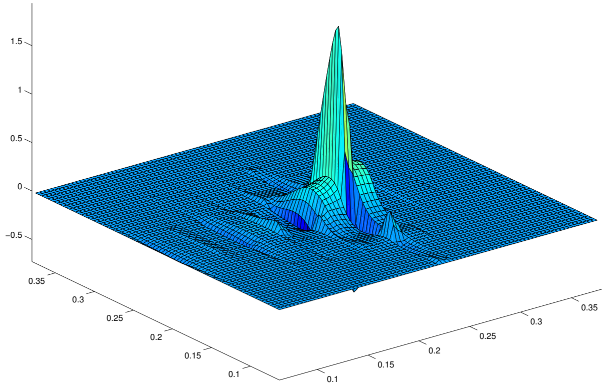
Example of corrected basis function

- With $\mathbf{b} = -[4, 0]'$.



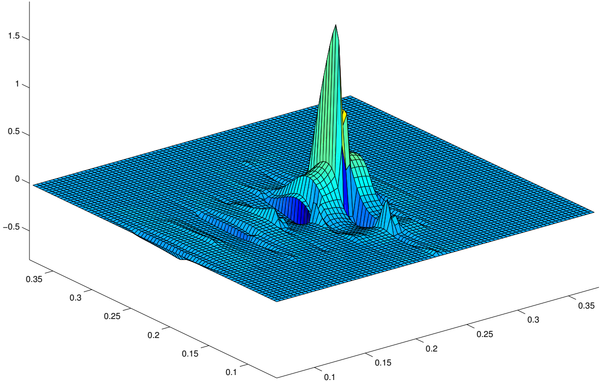
Example of corrected basis function

- With $\mathbf{b} = -[8, 0]'$.



Example of corrected basis function

- With $\mathbf{b} = -[16, 0]'$.



Discontinuous Galerkin multiscale method

Consider the problem: find $u_H^{ms,L} \in \mathcal{V}_L^{ms} = \text{span}\{\lambda_{T,j} - \phi_{T,j}^L\}$ such that

$$a_h(u_H^{ms,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H^{ms,L}.$$

- $\text{dim}\mathcal{V}_H^{ms,L} = \text{dim}\mathcal{V}_H$
- The basis function are solved independently of each other.
- Method can take advantage of periodicity.

A priori error bound for Poisson's equation with variable coefficients

Lemma (Decay of corrected basisfunctions)

For $\phi_{T,j} \in \mathcal{V}^f(\omega_i^L)$, there exist a, $0 < \gamma < 1$, such that

$$|||\phi_{T,j} - \phi_{T,j}^L||| \lesssim \gamma^L |||\lambda_j - \phi_{T,j}|||.$$

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$, there exist a, $0 < \gamma < 1$, such that

$$|||u - u_H^{ms,L}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L ||f||_{L^2}.$$

Choosing $L = \lceil C \log(H^{-1}) \rceil$ both terms behave in the same manor with an appropriate C.

Note: Theorem holds without any assumptions on scales or regularity!

Elfverson, Georgoulis, Målqvist and Peterseim
Convergence of discontinuous Galerkin multiscale methods. *Submitted.*

A priori error bound for convection-diffusion-reaction

Under the assumption $\mathcal{O}(\|A\|_{L^\infty(\Omega)}) = \mathcal{O}(\|H\mathbf{b}\|_{L^\infty(\Omega)})$ we have:

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Note: Theorem holds without any assumptions on scales or regularity!

Elfverson and Målqvist
Discontinuous Galerkin multiscale method for convection dominated problems. *Technical report.*

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$*, such that*

$$|||u - u_H^{ms,L}|||_h \leq |||u - u_H|||_h + C_{A_{max}/A_{min},f} H$$

given $f \in L^2(\Omega)$ *and choosing* $L = \lceil C \log(H^{-1}) \rceil$.

Note: Theorem holds without any assumptions on scales or regularity!

Poisson's equation on L-shaped domain

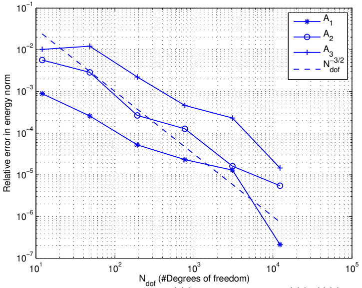

Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Choose $L = \lceil 2\log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5, 6\}$.
- Reference mesh is 2^{-8}.

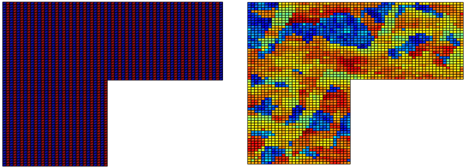

Figure: Permeabilities are piecewise constant on a mesh with size 2^{-5}, with ratio $A_{max}/A_{min} = \{10, 7 \cdot 10^6\}$

Convection-diffusion-reaction problems

$$-\nabla \cdot A \nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

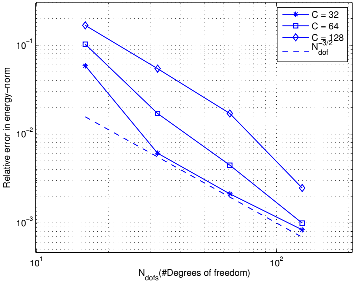

Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Let $A = 1$, $c = 0$, and $\mathbf{b} = C[1,0]'$ for $C = 32, 54, 128$.
- Choose $L = \lceil 2\log(\frac{1}{H})\rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{2,3,4,5\}$.
- Reference mesh is 2^{-7}.

Convection-diffusion-reaction problems

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $c = 0$, and $\mathbf{b} = [1, 0]'$.

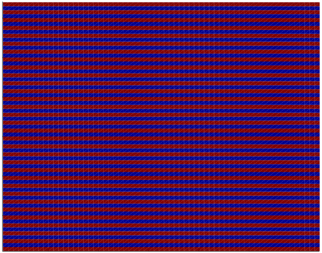

Figure: Diffusion coefficient A, $A_{max}/A_{min} = 100$ and $A_{min} = 0.01$.

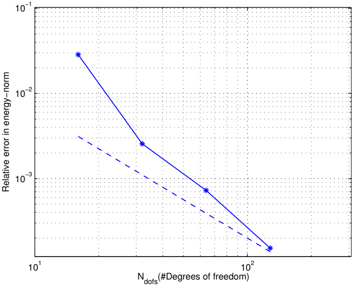

Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

Convection-diffusion-reaction problems

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $c = 0$, and $\mathbf{b} = [512, 0]'$.

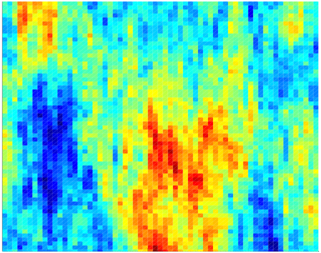

Figure: Diffusion coefficient A with $A_{max}/A_{min} \sim 10^5$.

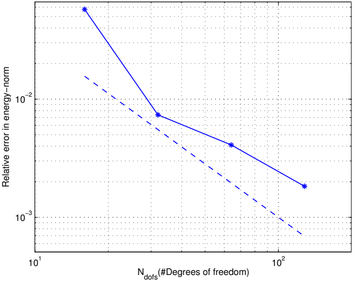

Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

Adaptivity and a posteriori error bound for pure diffusion

Theorem (A posteriori error bound)

Let $u_H^{ms,L}$ be the multiscale solution, then

$$|||u - u_H^{ms,L}||| \lesssim \left(\sum_{T \in \mathcal{T}_h} \rho_{h,T}^2 \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_h} \xi_{h,T}^2 \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_H} \rho_{L,\omega_T^L}^2 \right)^{1/2}.$$

- $\rho_{L,\omega_T^L}^2$, $\xi_{h,T}^2$ and $\rho_{h,K}^2$ depends on $u_H^{ms,L}$.
- $\rho_{L,\omega_T^L}^2$ measures the effect of the truncated patches.
- $\rho_{h,T}^2$ and $\xi_{h,T}^2$ measures the effect of the refinement level.

ELFVERSON, GEORGOULIS, AND MÅLQVIST
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *To appear in Multiscale Modeling and Simulations (MMS).*

Let $u_H^{ms,L} = \sum_{T\in\mathcal{T}} u_{H,T}^{ms,L}$ we have

$$\rho_{L,\omega_T^L}^2 = \sum_{e\in\Gamma(\partial\omega_T^L)} \frac{H^2}{hA_{min}} \left(\|\mathbf{n}\cdot\{A\nabla u_{H,T}^{ms,L}\}\|_{L^2(e)} + \frac{\sigma}{h}\|[u_{H,T}^{ms,L}]\|_{L^2(e)} \right)^2,$$

$$\rho_T = \frac{h}{A_{min}^{1/2}}\|(1-\Pi)f + \nabla\cdot A\nabla u_H^{ms,L}\|_{L^2(T)} + \frac{h^{1/2}}{A_{min}^{1/2}}\left(\|[A\nabla u_H^{ms,L}]\|_{L^2(\partial T)} + \frac{\sigma}{h}\|[u_H^{ms,L}]\|_{L^2(\partial T)} \right),$$

$$\xi_T^2 = \|A^{1/2}\nabla(u_H^{ms,L} - \mathcal{I}_h^c u_H^{ms,L})\|_{L^2(T)}^2 + \|\sqrt{\frac{\sigma}{h}}[u_H^{ms,L}]\|_{L^2(\partial T)}^2.$$

Numerical experiment

- We consider the permeabilities

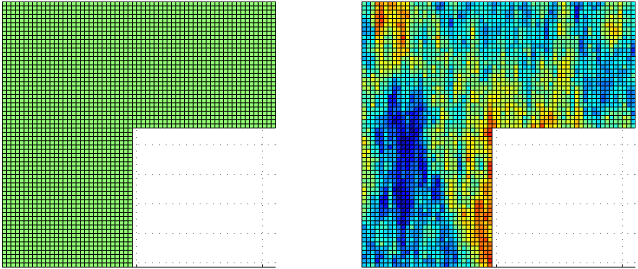

Figure: Permeabilities *One* left and *SPE* right.

Numerical experiments

- Using a refinement level of 30% we have.

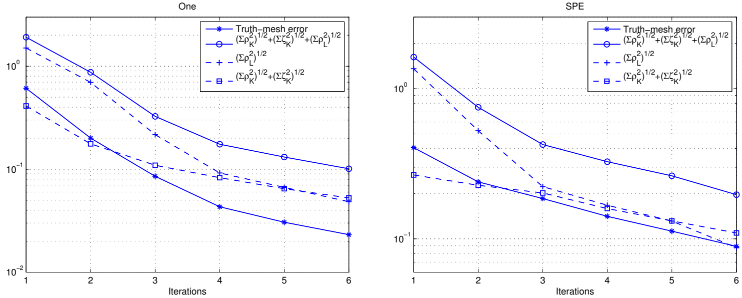

Figure: Convergence plot for *One* left and *SPE* right.

Numerical experiments

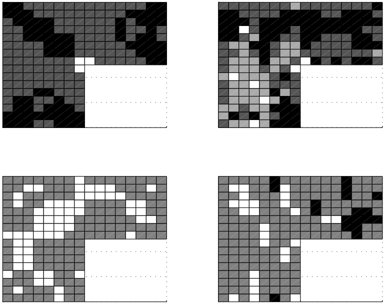

Figure: The level of refinement and size of the patches illustrated in the upper resp. lower plots for the different permeability One (left) and SPE (right). White is where most refinements resp. larger patch are used and black is where least refinements resp. smallest patches are used.

The End