UPPSALA UNIVERSITET

Discontinuous Galerkin multiscale methods

Daniel Elfverson
daniel.elfverson@it.uu.se

Division of Scientific Computing
Uppsala University
Sweden

Outline

1. Model problem and discretization
2. Discontinuous Galerkin Multiscale method
3. Convergence results
4. Adaptivity

D. Elfverson, G. H. Georgoulis, and A. Målqvist
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *Submitted.*

D. Elfverson, G. H. Georgoulis, A. Målqvist and D. Peterseim
Convergence of discontinuous Galerkin multiscale methods. *Submitted.*

Model problem

Consider the PDE

$$\begin{aligned} -\nabla \cdot A\nabla u &= f \text{ in } \Omega, \\ u &= 0 \text{ on } \partial\Omega. \end{aligned}$$

which in variational form reads, find $u \in \mathcal{V} := H^1_0(\Omega)$ such that

$$a(u, v) := \int_\Omega \nabla u \cdot \nabla v \, dx = \int_\Omega fv \, dx =: F(v) \quad \text{for all } v \in \mathcal{V}.$$

Discontinuous Galerkin discretization

- Split Ω into a elements $\mathcal{T} = \{T\}$, and let $\mathcal{E} = \{e\}$ be the set of all edges in $\mathcal{T}$.

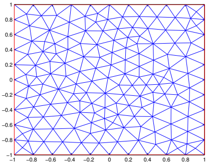


Figure: Example of a mesh on a unit square.

- Let $\mathcal{V}_h$ be the space of all discontinuous piecewise (bi)linear polynomials.

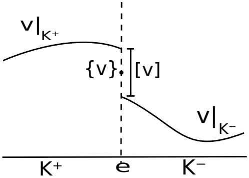


Figure: Example of $\{v\}$ and $[v]$

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

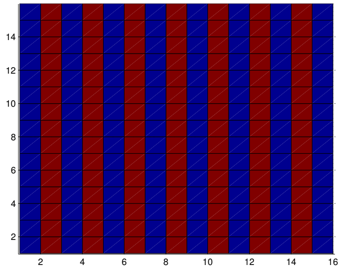


Figure: The coefficients A in the model problem.

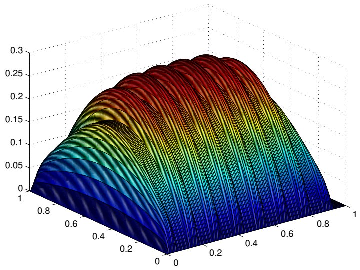


Figure: Reference solution.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

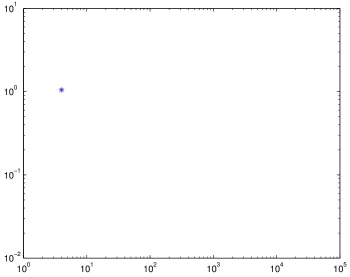


Figure: Energy norm with respect to the degrees of freedom.

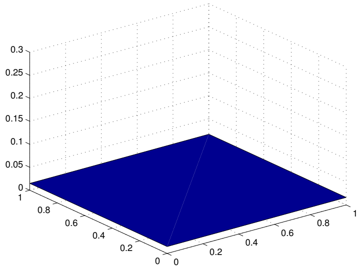


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

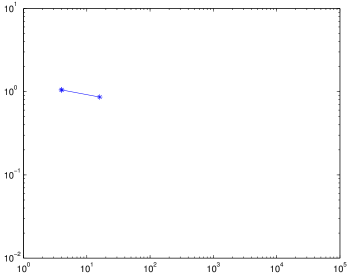


Figure: Energy norm with respect to the degrees of freedom.

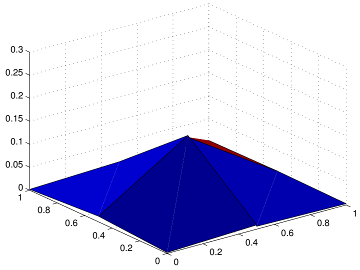


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

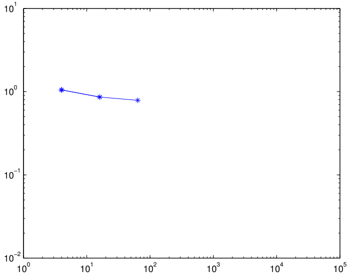


Figure: Energy norm with respect to the degrees of freedom.

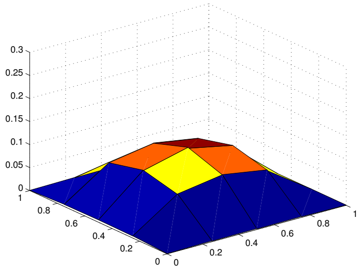


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

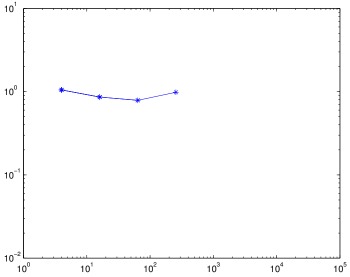


Figure: Energy norm with respect to the degrees of freedom.

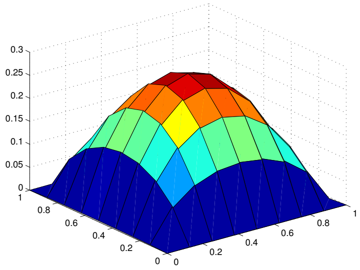


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

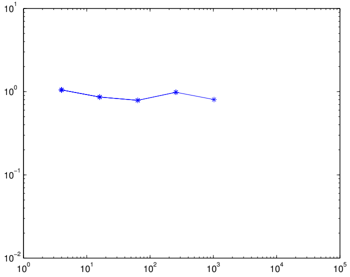


Figure: Energy norm with respect to the degrees of freedom.

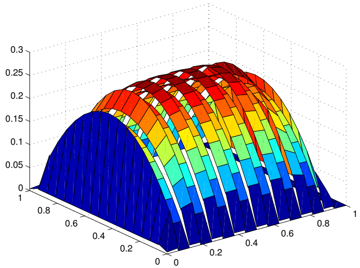


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

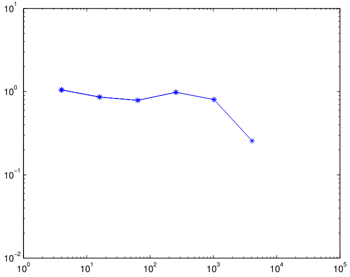


Figure: Energy norm with respect to the degrees of freedom.

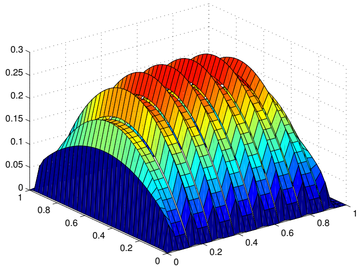


Figure: Solution obtained using the discontinuous Galerkin method.

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h.$$

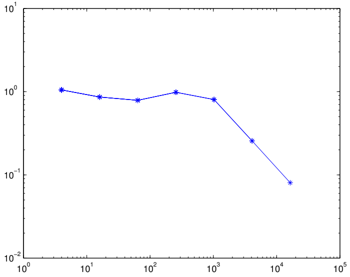


Figure: Energy norm with respect to the degrees of freedom.

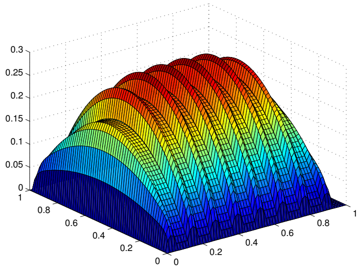


Figure: Solution obtained using the discontinuous Galerkin method.

Objective

- Eliminate the dependence of A via a multiscale method i.e.,

$$|||u - u_H^{ms,L}||| \leq C(f)H,$$

  where H does not resolve A
- Construct an adaptive algorithm to focus computational effort in critical areas.

Multiscale split

- Consider a coarse $\mathcal{V}_H$ and a fine mesh $\mathcal{V}_h$, such that $\mathcal{V}_H \subset \mathcal{V}_h$.
- Let Π_H be the L^2-projection onto $\mathcal{V}_H$. This will be used as the split between the coarse and fine scale.
- Define $\mathcal{V}^f(\omega) = \{v \in \mathcal{V}_h(\omega) : \Pi_H v = 0\}$.
- We have a L^2-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}^f$.

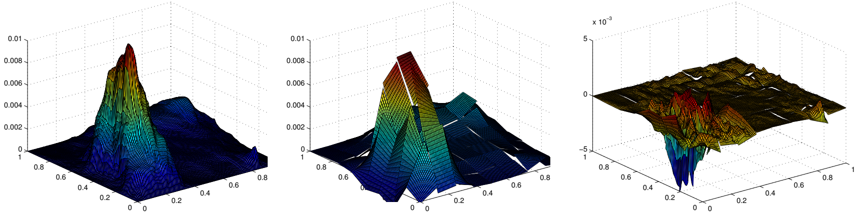


Figure: $u_h = u_H + u^f$

Corrected basis functions

- For each basis function $\lambda_{T,j} \in \mathcal{V}_H$ we calculate a corrector, find $\phi^L_{T,j} \in \mathcal{V}^f(\omega^L_T)$ such that

$$a_h(\phi^L_{T,j}, v_f) = a_h(\lambda_j, v_f), \quad \text{for all } v_f \in \mathcal{V}^f(\omega^L_T).$$

  where T is the element where $\phi^L_{T,j}$ lives, and L indicates the size of the patch.
- Let the new corrected space is defined by $\mathcal{V}^{ms}_H = \text{span}\{\lambda_j - \phi_j\}$.
- We have a_h-orthogonal split; $\mathcal{V}_h = \mathcal{V}^{ms}_H \oplus \mathcal{V}^f$

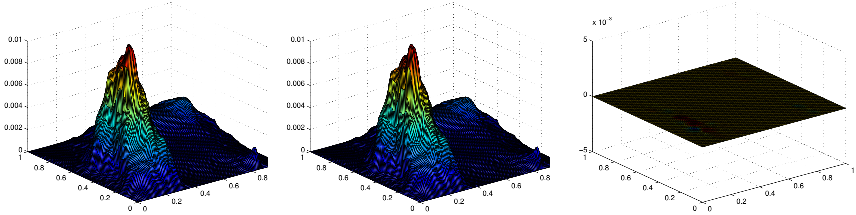


Figure: $u_h = u^{ms}_H + u^f$

Corrected basis functions

- For each basis function $\lambda_{T,j} \in \mathcal{V}_H$ we calculate a corrector, find $\phi^L_{T,j} \in \mathcal{V}^f(\omega^L_T)$ such that

$$a_h(\phi^L_{T,j}, v_f) = a_h(\lambda_j, v_f), \quad \text{for all } v_f \in \mathcal{V}^f(\omega^L_T).$$

  where T is the element where $\phi^L_{T,j}$ lives, and L indicates the size of the patch.
- Let the new corrected space is defined by $\mathcal{V}_H^{ms} = \text{span}\{\lambda_j - \phi_j\}$.
- We have a_h-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H^{ms} \oplus \mathcal{V}^f$

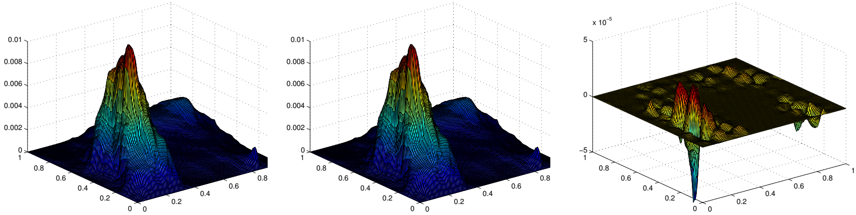
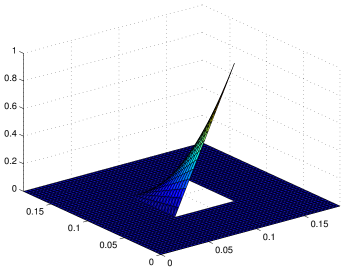
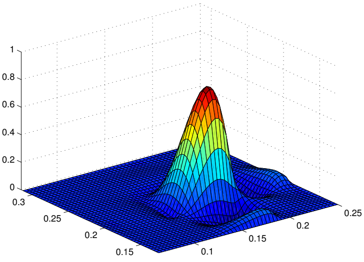
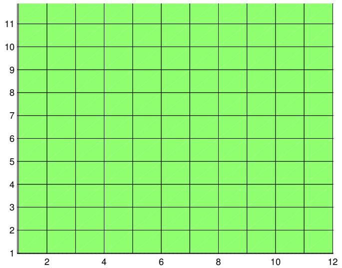


Figure: $u_h = u_H^{ms} + u^f$

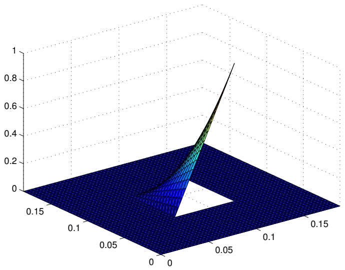
Examples of corrected basis functions

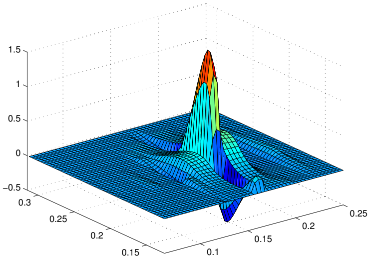


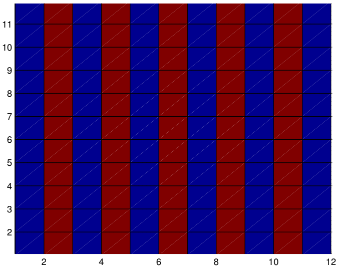




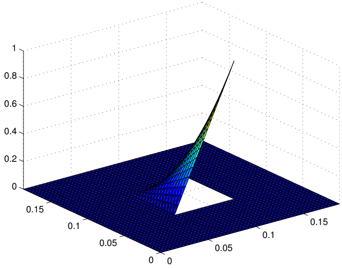
Examples of corrected basis functions

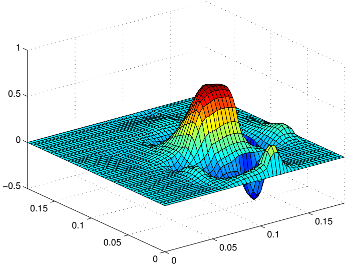


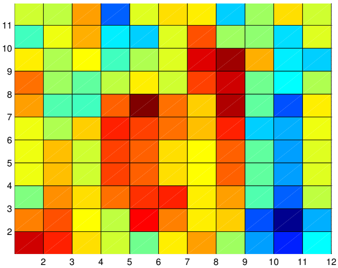




Examples of corrected basis functions







A priori results

Consider the problem: find $u_H^{ms,L} \in \mathcal{V}_L^{ms} = \text{span}\{\lambda_j - \phi_j\}$ such that

$$a_h(u_H^{ms,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H^{ms,L}.$$

Lemma (Decay of modifed basisfunction)

For $\phi_{T,j} \in \mathcal{V}^f(\omega_i^L)$, there exist a, $0 < \gamma < 1$, such that

$$|||\phi_{T,j} - \phi_{T,j}^L||| \lesssim \gamma^L |||\lambda_j - \phi_{T,j}|||.$$

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$, there exist a, $0 < \gamma < 1$, such that

$$|||u - u_H^{ms,L}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L ||f||_{L^2}.$$

Choosing $L = \lceil C \log(H^{-1}) \rceil$, then both terms behave in the same manor with a appropriate C.

Numerical verification

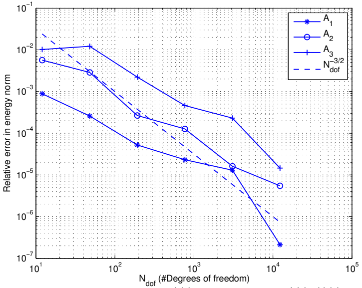


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Choose $L = \lceil 2\log(\frac{1}{H})\rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5, 6\}$.
- Reference mesh is 2^{-8}.

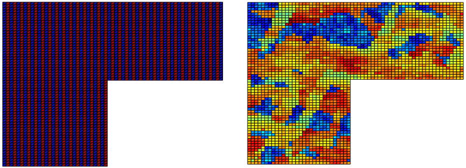


Figure: Permeabilities are piecewise constant on a mesh with size 2^{-5}, with ratio $\alpha_{max}/\alpha_{min} = \{10, 7 \cdot 10^6\}$

Adaptivity

- Construct an adaptive algorithm to automatically tune the fine mesh size and the patch sizes.
- We now consider a non-symmetric coarse scale problem, using local Neumann problems for the corrected basis functions, and using a right hand side correction.

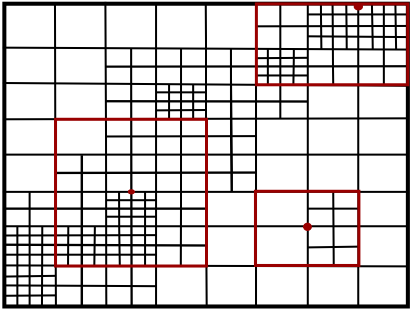


Figure: Example of an adapted mesh with varying patch sizes.

Theorem (A posteriori error estimate for ADG-MS)

Let $U_i := \sum_{j \in \mathcal{M}_i} U_j^{ms} + U_{f,i}$. *Then,*

$$|||u - U|||^2 \lesssim \sum_{T \in \mathcal{T}_h} \rho_{h,T}^2 + \sum_{T \in \mathcal{T}_H} \rho_{L,\omega_T^L}^2.$$

- $\rho_{L,\omega_i^L}^2$ measures the effect of the truncated patches.
- $\rho_{h,K}^2$ measures the effect of the refinement level.

Numerical experiment

- Refine 30% of the coarse elements or the patches are increase.

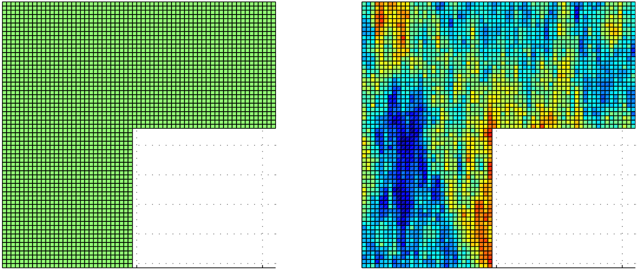


Figure: Permeabilities *One* left and *SPE* right.

Numerical experiments

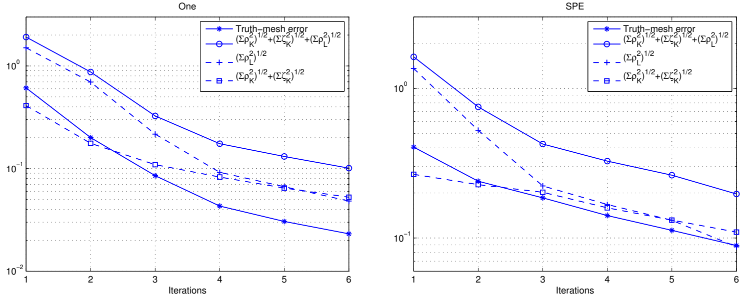


Figure: Convergence plot for *One* left and *SPE* right.

Numerical experiments

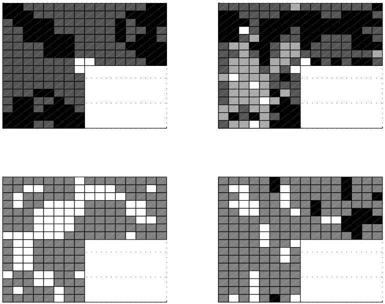


Figure: The level of refinement and size of the patches illustrated in the upper resp. lower plots for the different permeability One (left) and SPE (right). White is where most refinements resp. larger patch are used and black is where least refinements resp. smallest patches are used.

The End