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Discontinuous Galerkin multiscale methods for elliptic problems

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Outline

ELFVERSON, GEORGOULIS, AND MÅLQVIST
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *Submitted.*

ELFVERSON, GEORGOULIS, MÅLQVIST AND PETERSEIM
Convergence of discontinuous Galerkin multiscale methods. *In preparation.*

Model problem

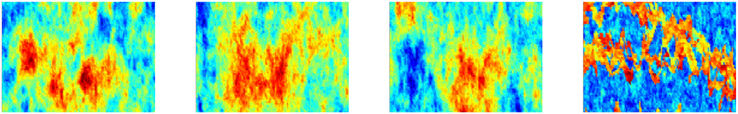
We seek the weak solution of Poisson's equation:

$$-\nabla \cdot A\nabla u = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \Gamma_D,$$
$$n \cdot A\nabla u = 0 \text{ on } \Gamma_N,$$

with $0 < \alpha \leq A(x) \leq \beta$ and $A \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $\int_\Omega f\, dx = 0$ if $\Gamma^D = 0$.

That is, find $u \in \mathcal{V} = \{v \in H^1(\Omega) : v|_{\Gamma_D} = 0 \text{ in the sense of traces}\}$ s.t.

$$a(u, v) := (A\nabla u, \nabla v) = (f, v) := F(v), \quad \text{for all } v \in \mathcal{V}.$$



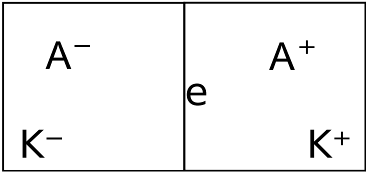
(a) $\beta/\alpha \sim 10^5$ (b) $\beta/\alpha \sim 10^5$ (c) $\beta/\alpha \sim 10^5$ (d) $\beta/\alpha \sim 10^6$

Figure: Permeabilities A projected in log scale and taken from the Society of Petroleum Engineer `http://www.spe.org/`.

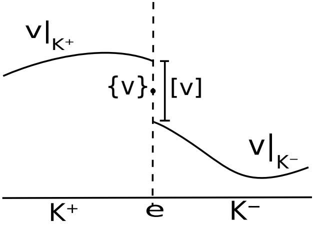
Discontinuous Galerkin discretization

- Consider the partition $\mathcal{K} = \{K\}$ and let $\mathcal{E}_\Omega$, $\mathcal{E}_{\Gamma_D}$, and $\mathcal{E}_{\Gamma_N}$ be the union of all, interior edges, edges on the Γ_D, resp. edges on Γ_N.
- Let also $\mathcal{V}_h$ be the space of all discontinuous piecewise (bi)linear polynomials.
- Define the weighted average and jump on face e as:

$$\{v\}_w = \frac{A^+ v^-}{A^+ + A^-} + \frac{A^- v^+}{A^+ + A^-} \quad \text{and} \quad [v] = v^+ - v^-.$$



(a) Here $\mathcal{K} = \{K^+, K^-\}$ and $\mathcal{E}_\Omega = \{e\}$



(b) Example of $\{v\}$ and $[v]$

Let

$$a_h(v,z) = \sum_{K\in\mathcal{K}} (A\nabla v, \nabla z)_{L^2(K)} - \sum_{e\in\mathcal{E}_\Omega\cup\mathcal{E}_{\Gamma_D}} \Big((\nu_e\cdot\{A\nabla v\}_w, [z])_{L^2(e)} + (\nu_e\cdot\{A\nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e\gamma_e}{h_e}([v],[z])_{L^2(e)} \Big),$$

$$F(v) = (f,v)_{L^2(\Omega)}.$$

where

$$|||v|||^2 = \sum_{K\in\mathcal{K}} \|\sqrt{A}\nabla v\|^2_{L^2(K)} + \sum_{e\in\mathcal{E}_\Omega\cup\mathcal{E}_{\Gamma_D}} \frac{\sigma_e\gamma_e}{h}\|[v]\|^2_{L^2(e)}$$

(One scale) DG method

Find $u_h \in \mathcal{V}_h$ such that

$$a_h(u_h, v) = F(v), \quad \text{for all } v\in\mathcal{V}_h.$$

Note: u_h will never be solved in practice, but it will act as a reference solution to compare the coarse grid approximation (multiscale solution) with.

Example

Let $A = A(x/\epsilon)$. We have the known result for periodic coefficients

$$|||u - u_H||| \leq C\frac{H}{\epsilon}||f||_{L^2(\Omega)}.$$

- Need $H < \epsilon$ for reliable results, computational prohibitive to solve on a single mesh.

Note: From now on we only consider $0 < \alpha \leq A(x) \in L^\infty(\Omega)$ without any assumptions on scale or periodicity.

Objective

- Eliminate the ϵ-dependence via a multiscale method i.e.,

$$|||u - u_H^{ms}||| \leq C(f)H.$$

- Construct an adaptive algorithm to focus computational effort in critical areas.

Some known methods

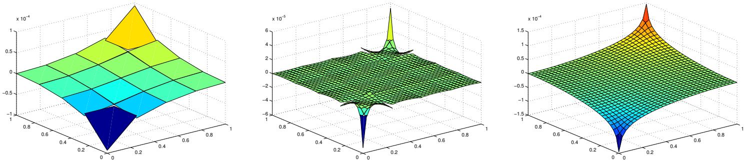
- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98.
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson-Målqvist 05, Nolen et al. 08, Nordbotten 09.
- Multiscale FEM: Hou-Wu 96, Efendiev-Ginting 04, Aarnes-Lie 06.
- Residual free bubbles: Brezzi et al. 98.
- Heterogeneous multiscale method: Engquist-E 03, E-Ming-Zang 04, Ohlberger 05.
- Equation free: Kevrekidis et al. 05.
- Metric based upscaling: Owhadi-Zang et al. 06.
- GFEM: Babuška et al. 94, Babuška-Lipton 11.

Remarks

- Local approximations (in parallel) on a fine scale are used to modify a coarse scale space or equation.
- Error analysis (for most methods) rely on strong assumptions such as scale separation and periodicity.

Variational multiscale framework

- Consider a coarse mesh $\mathcal{K}_H \subset \mathcal{K}_h$.
- Let $\mathcal{V}_H = span\{\phi_i\} = \Pi_H \mathcal{V}_h$ and $\mathcal{V}_f = \{v \in \mathcal{V}_h : \; \Pi_H v = 0\}$, where $\Pi_H : L^2 \to \mathcal{V}_H$ is the L^2 projection onto the coarse mesh.
- The problem is split into one coarse and fine scale contribution $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}_f$.



(c) u_H (d) u_h^f (e) $u_h = u_H + u_h^f$

Split $u_h = u_H + \mathcal{T} u_H + u_f$ and $v = v_H + v_f$ where $u_H, v_H \in \mathcal{V}_H$, $\mathcal{T} u_H, u_f, v_f \in \mathcal{V}_f$.

We obtain, find $u_H \in \mathcal{V}_H$ and $(\mathcal{T} u_H + u_f, v_H + v_f) \in \mathcal{V}_f$ s.t.

$$a_h(u_H + \mathcal{T} u_H + u_f, v_H + v_f) = l(v_H + v_f), \quad \forall v_H \in \mathcal{V}_H, v_f \in \mathcal{V}_f$$

Fine scale equation

Let $v_H = 0$ to get the fine scale equations

$$a_h(u_H + \mathcal{T} u_H, v_f) = F(v_f) - a_h(u_f, v_f),$$

split into two equations

$$a_h(u_f, v_f) = F(v_f) \quad \forall v_f \in \mathcal{V}_f,$$
$$a_h(\mathcal{T} u_H, v_f) = -a_h(u_H, v_f) \quad \forall v_f \in \mathcal{V}_f.$$

Note: Equally hard to solve as the original problem!

- Let $v_f = 0$ to get the coarse scale equations. Different chooses of coarse scale equations can be considered e.g.,

Non-symmetric coarse scale equation

$$a_h(u_H + \mathcal{T} u_H, v_H) = l(v_H) - a_h(u_f, v_H) \quad \forall v_H \in \mathcal{V}_H$$

Symmetric coarse scale equation

$$a_h(u_H + \mathcal{T} u_H, v_H + \mathcal{T} u_H) = l(v_H + \mathcal{T} u_H) - a_h(u_f, v_H + \mathcal{T} u_H) \quad \forall v_H \in \mathcal{V}_H$$

Symmetric coarse scale equation without correction term

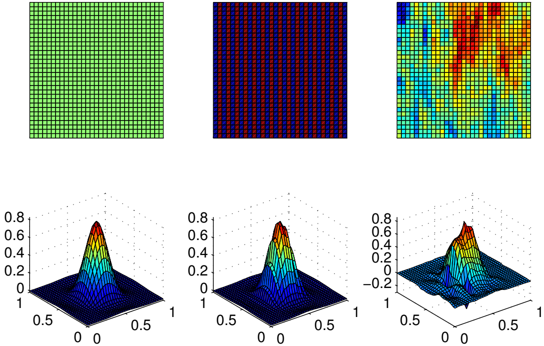
$$a_h(u_H + \mathcal{T} u_H, v_H + \mathcal{T} v_H) = l(v_H + \mathcal{T} v_H) \quad \forall v_H \in \mathcal{V}_H$$

View solution as span of corrected basis functions

- Recall the map $\mathcal{T} : \mathcal{V}_H \to \mathcal{V}_f$,

$$a_h(\mathcal{T} v_H, v_f) = -a_h(v_H, v_f), \quad \forall v_H \in \mathcal{V}_H, v_f \in \mathcal{V}_f.$$

- We let $\mathcal{V}^{ms} = \mathcal{V}_H + \mathcal{T}\mathcal{V}_H = \text{span}\{\phi_i + \mathcal{T}\phi_i\}$.
- $\phi_i + \mathcal{T}\phi_i$ can be viewed as a coarse modified basis function.
- From the multiscale map we have, $\mathcal{V}_h = \mathcal{V}_{ms} \oplus_a \mathcal{V}_f$.



Localization of $\mathcal{T}\phi_i$

- For each i we have, $a_h(\mathcal{T}_i^L\phi_i, v) = -a_h(\phi_i, v)$ for all $v \in \mathcal{V}_f(\omega_i^L)$, solved on local Dirichlet or Neumann patches.
- Define the localized multiscale space by, $\mathcal{V}_L^{ms} := \text{span}\{\phi_i + \mathcal{T}_i^L\phi_i\}$.

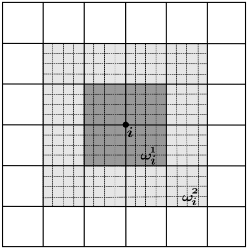


Figure: Example of a one layer patch ω_i^1 and a two layer patch ω_i^2

A priori results

Consider the problem: find $u_{H,L}^{ms} \in \mathcal{V}_L^{ms} = \mathrm{span}\{\phi_i + \mathcal{T}_i^L \phi_i\}$ such that

$$a_h(u_{H,L}^{ms}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_L^{ms},$$

where local Dirichlet patches has been used for the corrected basis functions.

Lemma (Decay of corrected basis function)

For $\mathcal{T}_i^L \phi_i \in \mathcal{V}_f(\omega_i^L)$, there exist a, $0 < \gamma < 1$, such that

$$|||\mathcal{T}\phi_i - \mathcal{T}_i^L \phi_i||| \lesssim \gamma^L |||\phi_i + \mathcal{T}\phi_i|||_{\omega_i^L}.$$

Theorem

For $u_{H,L}^{ms} \in \mathcal{V}_L^{ms}$, there exist a, $0 < \gamma < 1$, such that

$$|||u - u_{H,L}^{ms}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L ||f||_{L^2}.$$

Note: Theorem holds without any assumptions on scales or regularity!

Decay of corrected basisfunction

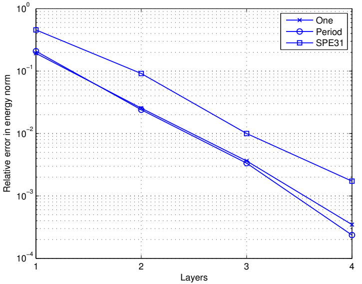
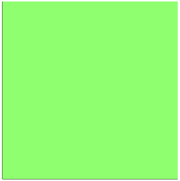
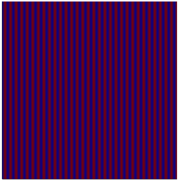
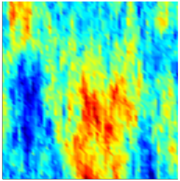


Figure: $\#Layers$ vs $|||\mathcal{T}\phi_i - \mathcal{T}_i^L\phi_i|||/|||\mathcal{T}\phi_i|||$

- Let the computational domain be ω_i^L for $L = 1, 2, \ldots, N$ where $\omega_i^L \subseteq \Omega$.
- Coarse mesh is 8×8 element and reference grid is 64×64 elements.







Numerical verification

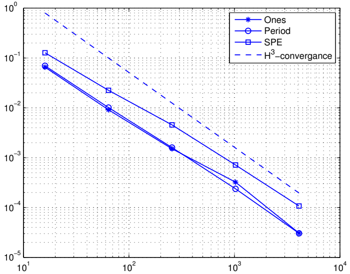


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Choose $L = \lceil 2\log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5\}$.
- Reference mesh is 2^{-7}.

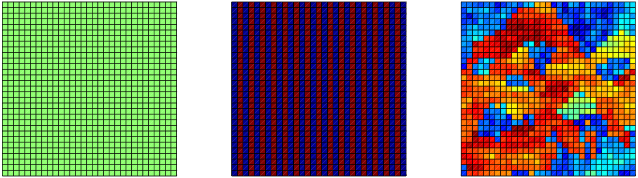


Figure: Permeabilities are piecewise constant on a mesh with size 2^{-5}, with ratio $\beta/\alpha = \{1, 10, 7 \cdot 10^6\}$

Adaptivity

- Construct an adaptive algorithm to automatically tune the fine mesh size and the patch sizes.
- We now consider a non-symmetric coarse scale problem, using local Neumann problems for the corrected basis functions, and using a right hand side correction.

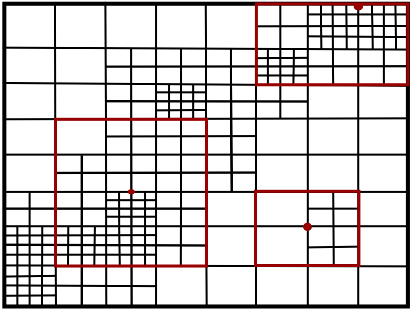


Figure: Example of an adapted mesh with varying patch sizes.

For the bilinear form to be well defined for low regular solutions, let $a_h : (\mathcal{V} + V_h) \times (\mathcal{V} + V_h) \to \mathbb{R}$ be defined as

$$a_h(v,z) = \sum_{K \in \mathcal{K}} (A\nabla v, \nabla z)_{L^2(K)} - \sum_{e \in \mathcal{E}_\Omega \cup \mathcal{E}_{\Gamma_D}} \Big((\nu_e \cdot \{A\mathcal{P}\nabla v\}_w, [z])_{L^2(e)} + (\nu_e \cdot \{A\mathcal{P}\nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e \gamma_e}{h_e}([v],[z])_{L^2(e)} \Big).$$

where $\mathcal{P} : (L^2)^d \to (\mathcal{V}_h)^d$ is the L^2-projection onto $\mathcal{V}_h$.

Note: a_h is no longer consistent, but the consistency error will decrease as $h \to 0$.

- Let $\mathcal{N}$ be the set of all coarse nodes and $\mathcal{M}_i$ be the set of all j such that $\phi_j(x_i) = 1$.
- Let $\tilde{\mathcal{V}}^{ms} = \text{span}\{\phi_j + \mathcal{T}_i^{L(i)}\phi_j\}$, with varying patch sizes.
- Let $U_h^f = \sum_{i\in\mathcal{N}} U_{h,i}^f$ be a right hand side correction obtained by solving: find $U_{h,i}^f \in \mathcal{V}_f(\omega_i^{L(i)})$ such that

$$a_h(U_{h,i}^f, v) = F(\Phi_i v), \quad \text{for all } v \in \mathcal{V}_f(\omega_i^{L(i)}).$$

Coarse equation (with right hand side correction)

We consider: find $U^{ms} \in \tilde{\mathcal{V}}^{ms}$ such that

$$a_h(U^{ms}, v) = F(v) - a_h(U_h^f, v), \quad \text{for all } v \in \mathcal{V}_H.$$

where the multiscale solution is $U = U^{ms} + U_h^f$.

Convergence

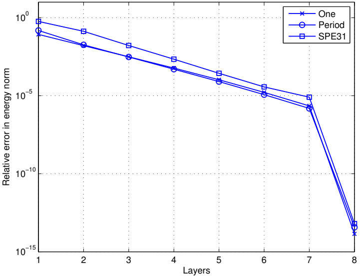
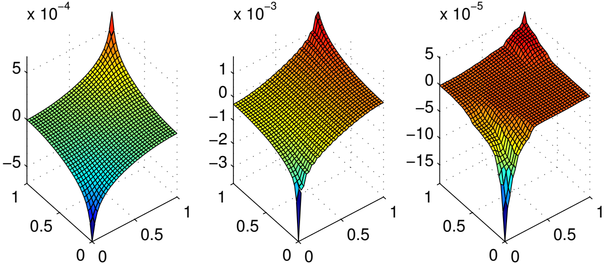


Figure: $\#Layers$ vs $|||U_{ref} - U|||/|||U_{ref}|||$

- The coarse grid is 8×8 coarse elements.
- The reference solution U_{ref} is the DG solution computed on 64×64 elements.
- The right hand side is -1 in the lower left corner and 1 in the upper right.



Theorem (A posteriori error estimate for DG method)

- *Let u, u_h be given by the exact solution respectively the DG solution.*
- *Let $\chi = \mathcal{I}^c u_h$ where $\mathcal{I}^c : \mathcal{V}_h \to \mathcal{V}_h \cap H^1$ is a averaging interpolation operator.*
- *Moreover, let $\mathfrak{E} := \mathfrak{E}_c + \mathfrak{E}_d$ where $\mathfrak{E}_c := u - \chi$ and $\mathfrak{E}_d := \chi - u_h$.*

Then,

$$\sum_{K \in \mathcal{K}} ||\sqrt{A}\nabla\mathfrak{E}||^2_{L^2(K} \lesssim \sum_{K \in \mathcal{K}} (\varrho_K(u_h) + \zeta_K(u_h, \chi))^2,$$

where

$$\varrho_K(u_h) = \frac{h_K}{\sqrt{\alpha}}||f + \nabla \cdot A\nabla u_h||_{L^2(K)}, + \sqrt{\frac{h_K}{\alpha}}\Big(||(1 - w_{K(e)})n \cdot [A\nabla u_h]||_{L^2(\partial K)} + ||\frac{\sigma_e \gamma_e}{h_e}[u_h]||_{L^2(\partial K \setminus \Gamma^B)}\Big),$$

$$\zeta_K(u_h, \chi) = ||\sqrt{A}\nabla(u_h - \chi)||_{L^2(K)}.$$

Lemma (Averaging interpolation operator)

- *Let* $\mathcal{I}^c : \mathcal{V}_h \to \mathcal{V}_h \cap H^1$

$$\mathcal{I}^c v = \sum_{i \in \mathcal{N}} \Big(\frac{1}{|\mathcal{M}_j|} \sum_{j \in \mathcal{M}_i} v_j(x_i) \Phi_j \Big).$$

Then,

$$||\sqrt{A} \nabla (v - \mathcal{I}^c v)||^2 \lesssim \beta || \frac{1}{\sqrt{h_e}} [v] ||^2_{L^2(\partial K \setminus \mathcal{E}_{\Gamma^B})}.$$

- Under certain assumption on A, $\zeta_K(u_h, \chi)$ can be estimated and hidden in $\varrho_K(u_h)$.

Theorem (A posteriori error estimate for ADG-MS)

- *Let u, U be the exact solution respectively the multiscale solution.*
- *Let $X = \mathcal{I}^c U \in H^1(\Omega)$.*
- *Set $\mathscr{E} := \mathscr{E}_c + \mathscr{E}_d$ where $\mathscr{E}_c := u - X$ and $\mathscr{E}_d := X - U$.*
- $U_i := \sum_{j\in\mathcal{M}_i} U_{c,j}(\phi_j + \tilde{\mathcal{T}}\phi_j) + U_{f,i}$*, where $U_{c,j}$ are the nodal values.*

Then,

$$|||\sqrt{A}\nabla\mathscr{E}|||^2 \lesssim \sum_{K_c\in\mathcal{K}_c} \rho^2_{h,K_c} + \sum_{i\in\mathcal{N}} \rho^2_{L,\omega_i^L},$$

where

$$\rho^2_{L,\omega_i^L} = \sum_{e\in\mathcal{E}_{\Gamma B}(\omega_i^L)\setminus\mathcal{E}_{\Gamma B}} \rho^2_{L,\omega_i^L,e},$$

$$\rho_{L,\omega_i^L,e} = \frac{H_{\omega_i^L}}{\sqrt{h_K\alpha}}\Big(||n\cdot\{A\nabla U_i\}_w||_{L^2(e)} + \frac{\sigma_e\gamma_e}{h_e}||[U_i]||_{L^2(e)}\Big),$$

measures the effect of the truncated patches.

Also

$$\rho_{h,K_c}^2 = \sum_{K \in K_c} (\varrho_K(U) + \zeta_K(U,X))^2,$$

with ϱ_K and ζ_K as in previous theorem.

Comments

- $\rho_{L,\omega_i^L}^2$ measure the effect of the truncated patches.
- $\rho_{h,K}^2$ measure the effect of the refinement level.
- $\frac{1}{\sqrt{h_K}h_e}|||[U_i]|||_{L^2(e)}$ behave as $h^{-3/2}e^{-L} \sim 1 \Rightarrow L \sim \frac{3}{2}\log h^{-1}$

Sketch of proof

We have

$$a(\mathscr{E}_c, \mathscr{E}_c) = a(\mathscr{E}, \mathscr{E}_c) - a(\mathscr{E}_d, \mathscr{E}_c),$$

where

$$\begin{aligned} a(\mathscr{E}, \mathscr{E}_c) &= a(u, \mathscr{E}_c) - a(U, \mathscr{E}_c), \\ &= F(\mathscr{E}_c) - a(U, \mathscr{E}_c), \\ &= F(\mathscr{E}_c - v_c) - a(U, \mathscr{E}_c - v_c), \\ &= \sum_{i \in \mathcal{N}} \Big(F(\Phi_i(\mathscr{E}_c - v_c - v_f)) - a(U_i, \mathscr{E}_c - v_c) + a_i(U_i, v_f) \Big). \end{aligned}$$

here $v_c \in \mathcal{V}_c$ and $v_f \in \mathcal{V}_f$.
Notice that

$$\begin{aligned} a_i(\mathscr{E}, \mathscr{E}_c) = a(\mathscr{E}, \mathscr{E}_c) + \sum_{e \in \mathcal{E}_{\Gamma B}(\omega_i^L) \setminus \mathcal{E}_{\Gamma B}} \Big(&(n \cdot \{A \nabla U_i\}_w, [v_f])_{L^2(e)} \\ &+ (n \cdot \{A \nabla U_i\}_w, [v_f])_{L^2(e)} - \frac{\sigma_e \gamma_e}{h_e}([U_i], [v_f])_{L^2(e)} \Big) \end{aligned}$$

Sketch of proof

Then,

$$
\begin{aligned}
a(\mathscr{E}, \mathscr{E}_c) &= \Big(F(\mathscr{E}_c - v_c - v_f) - a(\mathscr{E}, \mathscr{E}_c - v_c - v_f) \Big) \\
&+ \sum_{i \in \mathcal{N}} \sum_{e \in \mathcal{E}_{\Gamma B}(\omega_i^L) \setminus \mathcal{E}_{\Gamma B}} \Big((n \cdot \{A \nabla U_i\}_w, [v_f])_{L^2(e)} + (n \cdot \{A \nabla U_i\}_w, [v_f])_{L^2(e)} \\
&- \frac{\sigma_e \gamma_e}{h_e} ([U_i], [v_f])_{L^2(e)} \Big) \\
&=: I + II.
\end{aligned}
$$

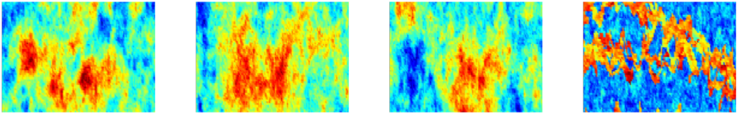
1. The first term (I), is bounded by the a a posteriori error estimate for DG.
2. To bound the second term (II),
   - Select v_c and v_f as the piecewise constant L^2-projection onto V_c and V_f, respectively.
   - Then using a trace inequality, a interpolation estimate and L^2-stability of π_f, II is bounded.

Algorithm 1 Adaptive Discontinuous Galerkin Multiscale Method

Initialize the coarse mesh with mesh size H.
Let the fine mesh size be $h_K = H/4$ for all $K \in \mathcal{K}_H$ and $L(\omega_i) = 2$ for all $i \in \mathcal{N}$
while $\sum_{i \in \mathcal{N}} (\rho^2_{h,\omega_i} + \rho^2_{L,\omega_i}) > TOL$ **do**
  for $i \in \mathcal{N}$ **do**
    if $\rho^2_{L,\omega_i} > TOL/(2\mathcal{N})$ **then**
      $L(\omega_i) := L(\omega_i) + 1$
    end if
  end for
  for $K_H \in \mathcal{K}_H$ **do**
    if $\rho^2_{h,K} > TOL/(2|\mathcal{K}_H|)$ **then**
      $h_K := h_K/2$
    end if
  end for
end while

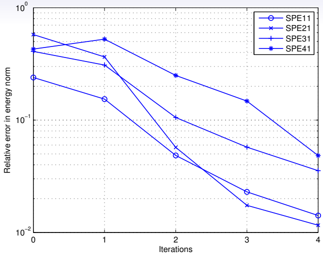
Numerical experiment

- Refine 30% of the coarse elements and increase 30% of the patch sizes in each iteration.
- Coarse mesh is 32×32 elements and reference grid is 256×256 elements.
- The right hand side is -1 in the lower left corner and 1 in the upper right.

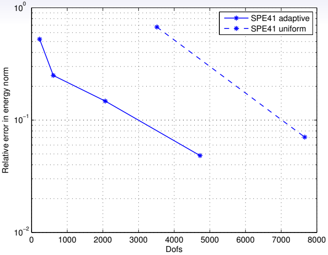


(a) $\beta/\alpha \sim 10^5$ (b) $\beta/\alpha \sim 10^5$ (c) $\beta/\alpha \sim 10^5$ (d) $\beta/\alpha \sim 10^6$

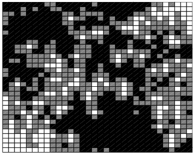
Figure: Permeabilities A projection in log scale.



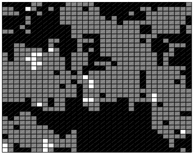
(a) The relative error in broken energy norm with respect to number of iterations. Iteration 0 corresponds to the standard DG solution and iteration 1 the start values in the adaptive algorithm.



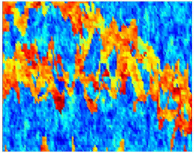
(b) The relative error in broken energy norm with respect to the mean value of the degrees of freedom for the fine scale problems.



(c) Refinement level, h_K



(d) Layers, L



(e) Permeability, A

Conclusions:

- The fine scale problems are perfectly parallelizable.
- The exponential decay in the corrected basis function allows small patches.
- The error estimate and the adaptivity algorithm focus computational effort in critical areas.
- Get optimal convergence for the (crude) SPE Benchmark problem.
- DG: Flexibility in fine scale approximation spaces, boundary conditions and good conservation properties of the state variable

Future work

- Using DG on the coarse scale but CG on the fine scale to save computational work.
- Extend analysis to convection-diffusion problems.
- Construct an adaptive algorithm that increases the patch sizes only in the direction where the error is large.

Questions