A discontinuous Galerkin local orthogonal decomposition (LOD) method for elliptic multiscale problems

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Applications of multiscale methods

- Subsurface flow
- Composite materials
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Need numerical solution of partial differential equations with rough data (module of elasticity, conductivity, permeability, etc)

Major challenge

Solution has features on a several non-seperal scales
Model problem

Consider the elliptic model problem

\[-\nabla \cdot A \nabla u + (b \cdot \nabla u) = f \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial \Omega,\]

where we assume:

- \(0 < A_{\text{min}} \in \mathbb{R} \leq A(x) \in L^\infty(\Omega, \mathbb{R}^{d \times d}_{\text{sym}})\)
- \(f \in L^2(\Omega)\)
- \(b \in [W^1_\infty(\Omega)]^d\) and \(\nabla \cdot b = 0\)
Discontinuous Galerkin discretization

- Split $\Omega$ into elements $\mathcal{T} = \{T\}$, and let $\mathcal{E} = \{e\}$ be the set of all edges in $\mathcal{T}$.

- Let $\mathcal{V}_H$ be the space of all discontinuous piecewise (bi)linear polynomials.

![Example of a mesh on a unit square.](image)

![Example of $\{v\}$ and $[v]$](image)
The bilinear form is defined by:

\[ a_h(u, v) := a_h^d(u, v) + a_h^{c-r}(u, v). \]

where

\[
\begin{align*}
a_h^d(u, v) &:= (A\nabla_h u, \nabla_h v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_h} \left( \frac{\sigma_e}{h_e} ([u], [v])_{L^2(e)} - ([\nu_e \cdot A\nabla u], [v])_{L^2(e)} - ([\nu_e \cdot A\nabla v], [u])_{L^2(e)} \right), \\
a_h^{c-r}(u, v) &:= (b \cdot \nabla_h u + cu, v)_{L^2(\Omega)} + \sum_{e\in\mathcal{E}_h} (b_e [u], [v])_{L^2(e)} \\
&- \sum_{e\in\mathcal{E}_h(\Omega)} ([\nu_e \cdot b] u, [v])_{L^2(e)} - \sum_{e\in\mathcal{E}_h(\Gamma)} \frac{1}{2} ([\nu_e \cdot b] u, v)_{L^2(e)},
\end{align*}
\]

where \( \sigma_e \) is a constant and

\[ b_e = |\nu_e \cdot b|/2. \]

- \( a_h^d(\cdot, \cdot) \) approximates the diffusion a interior penalty method.
- \( a_h^{c-r}(\cdot, \cdot) \) approximates the convection-reaction using upwind.
Discontinuous Galerkin discretization

- \( a_h(\cdot, \cdot) \): symmetric interior penalty (SIPG) and upwind.
- The energy-norm is defined by

\[
\|\cdot\|_{h}^2 = \|A^{1/2} \nabla H \cdot \|_{L^2(\Omega)}^2 + \sum_{e \in \mathcal{E}} \left( \frac{\sigma}{H} + \frac{|b \cdot n|^2}{2} \right) \|\cdot\|_{L^2(e)}^2
\]

(One scale) DG method

Find \( u_h \in \mathcal{V}_h \) such that

\[ a_h(u_h, v) = F(v), \quad \text{for all } v \in \mathcal{V}_h. \]
(One scale) DG method ($b = 0$)

Find $u_H \in \mathcal{V}_H$ such that

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Figure: Energy norm with respect to the degrees of freedom.

Figure: Solution obtained using the discontinuous Galerkin method.
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Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$
(One scale) DG method \((b = 0)\)

Find \(u_H \in V_H\) such that

\[
a_H(u_H, v) = F(v), \quad \text{for all } v \in V_H.
\]

**Figure:** Energy norm with respect to the degrees of freedom.

**Figure:** Solution obtained using the discontinuous Galerkin method.
Objective with the multiscale method

- Eliminate the dependency of $A$ via a multiscale method i.e.,

\[ |||u - u_{H}^{\text{ms,L}}||| \leq C_f H, \]

where $H$ does not resolve the variation in $A$

- Construct an adaptive algorithm to focus computational effort to critical areas (for the case with pure diffusion)
Incomplete list of other multiscale methods

- Variational multiscale method (VMS): [Hughes et al. 95]
- Multiscale FEM (MsFEM): [Hou-Wu 96]
- Heterogeneous multiscale method (HMM): [Engquist, E 03]
- Multiscale finite volume method: [Jenny et al. 03]
- Residual free bubbles: [Brezzi et al. 98]
- Upscaling techniques: [Durlofsky et al. 98]
- Equation free: [Kevrekidis et al. 05]
- Metric based upscaling: [Owhadi-Zang 06]
- Polyharmonic homogenization [Owhadi-Zang 12]
- Generalised MsFEM [Efendiev et al. 10]
- Mortar Multiscale Methods [Arbogast et al, 07]
- ...
Local orthogonal decomposition

- Adaptivity [Larson, Målqvist 07], [Målqvist 11]
- Convergence analysis [Målqvist, Peterseim 14]
- Convergence analysis for DG [Elfverson et al. 13]
- Convection problem [Submitted]
- Semi-linear elliptic problem [Henning et al. 14]
- Eigenvalue problem [Målqvist, Peterseim 14]
- Non-linear Schrödinger equation [Henning et al. 14]
- Petrov-Galerkin formulation [Submitted]
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Remarks

- Builds on the idea of VMS
- Error analysis DOESN’T rely on assumptions such as scale separation and periodicity
- Error analysis does depend on the contrast, however numerical test show a very weak dependence
Local orthogonal decomposition

- Adaptivity [Larson, Målqvist 07], [Målqvist 11]
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Multiscale split

- Consider $\mathcal{V}_H$ and $\mathcal{V}_h$, such that $\mathcal{V}_H \subset \mathcal{V}_h$.
- Let $\Pi_H$ be the $L^2$-projection onto $\mathcal{V}_H$.
- Define $\mathcal{V}^f(\omega) = \{ v \in \mathcal{V}_h(\omega) : \Pi_H v = 0 \}$.
- We have a $L^2$-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}^f$.

Figure: $u_h = u_H + u^f$
Corrected basis functions

• For each $\lambda_{T,j} \in \mathcal{V}_H$ we compute a corrector, find $\phi_{T,j}^L \in \mathcal{V}_f(\omega_T^L)$ such that

$$a_h(\phi_{T,j}^L, v_f) = a_h(\lambda_{T,j}, v_f), \quad \text{for all } v_f \in \mathcal{V}_f(\omega_T^L).$$

where $L$ indicates the size of the patch.

• Corrected space: $\mathcal{V}_H^{ms} = \text{span}\{\lambda_{T,j} - \phi_{T,j}^L\}$.

• We have a $a(\cdot, \cdot)$-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H^{ms} \oplus \mathcal{V}_f$.

Figure: $u_h = u_H^{ms} + u^f$
Mesh patch
Examples of corrected basis functions
Examples of corrected basis functions
Examples of corrected basis functions
Example of corrected basis function

- With $b = [0, 0]'$. 

![Graph showing coefficients $A$ in the model problem.](image-url)
Example of corrected basis function

- With $\mathbf{b} = -[1, 0]'$.
Example of corrected basis function

- With $b = -[2, 0]'$. 

![Graph showing the corrected basis function with $b = -[2, 0]'$.]
Example of corrected basis function

- With $b = [-4, 0]$.
Example of corrected basis function

- With $b = -[8, 0]'$. 

![3D plot of corrected basis function](image-url)
Example of corrected basis function

- With $b = -[16, 0]'$. 
Discontinuous Galerkin multiscale method

Consider the problem: find $u_{H}^{ms,L} \in \mathcal{V}_{H}^{ms,L} = \text{span}\{\lambda_{T,j} - \phi_{T,j}^{L}\}$ such that

$$a_{h}(u_{H}^{ms,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_{H}^{ms,L}.$$  

- $\dim\mathcal{V}_{H}^{ms,L} = \dim\mathcal{V}_{H}$
- The basis function are solved independently of each other.
- Method can take advantage of periodicity.
A priori error bound

Under the assumption $O(\|Hb\|_{L^\infty(\Omega)}/A_{\min}) = 1$ it holds:

**Lemma (Decay of corrected basisfunction)**

For $\phi_{T,j} \in V^f(\omega^L_j)$, there exist $a$, $0 < \gamma < 1$, such that

$$|||\phi_{T,j} - \phi_{T,j}^L||| \lesssim \gamma^L |||\lambda_j - \phi_{T,j}|||.$$  

**Theorem**

For $u_{H^L}^{ms} \in V_{H^L}^{ms}$, there exist $a$, $0 < \gamma < 1$, such that

$$|||u - u_{H^L}^{ms}||| \lesssim |||u - u_h||| + |||H(f - \Pi_Hf)|||_{L^2} + H^{-1}(L)^{d/2} \gamma^L |||f|||_{L^2}.$$  

Choosing $L = \lceil C \log(H^{-1}) \rceil$ both terms behave in the same manor with an appropriate $C$. 


**Pure diffusion on L-shaped domain**

- Choose $L = \lceil 2 \log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + \sin(\pi x) + \sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5, 6\}$.
- Reference mesh is $2^{-8}$.

![Figure: #dofs vs $|| u_h - u_{H,L}^{ms} || / || u_h ||$](image1)

![Figure: Permeabilities are piecewise constant on a mesh with size $2^{-5}$, with ratio $A_{max}/A_{min} = \{10, 7 \cdot 10^6\}$](image2)
Numerical verification of the convergence

\[-\nabla \cdot A \nabla u + b \cdot \nabla u = f \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial \Omega.\]

- Let $A = 1$ and $b = C[1, 0]'$ for $C = 32, 54, 128$.
- Choose $L = \lceil 2 \log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + \sin(\pi x) + \sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{2, 3, 4, 5\}$.
- Reference mesh is $2^{-7}$.

Figure: $\#\text{dofs} \text{ vs } \|\|u_h - u_{H,L}^{ms}\|\|/\|\|u_h\||$
- $\nabla \cdot A \nabla u + b \cdot \nabla u = f$ in $\Omega$, 
$u = 0$ on $\partial \Omega$.

- Let $b = [1, 0]'$.

Figure: Diffusion coefficient $A$, $A_{\text{max}}/A_{\text{min}} = 100$ and $A_{\text{min}} = 0.01$.

Figure: \#dofs vs $\|u_h - u_{H,s}^m\|/\|u_h\|$
\[-\nabla \cdot A \nabla u + b \cdot \nabla u = f \text{ in } \Omega,\]
\[u = 0 \text{ on } \partial \Omega.\]

- Let \(b = [512, 0]'\).

**Figure**: Diffusion coefficient \(A\) with \(A_{\text{max}}/A_{\text{min}} \sim 10^5\) and \(A_{\text{min}} = 0.05\).

**Figure**: \#dofs vs \(\|u_h - u_{H,L}^{\text{ms}}\|/\|u_h\|\)
Petrov-Galerkin DG-LOD

Consider the problem: find \( u_{ms,L}^H \in V_{ms,L}^H = \text{span}\{\lambda_{T,j} - \phi_{T,j}^L\} \) such that

\[
a_h(u_{ms,L}^H, v) = F(v), \quad \text{for all } v \in V_H = \text{span}\{\lambda_{T,j}\}
\]

Same as before:

- \( \dim V_{ms,L}^H = \dim V_H \)
- The basis function are solved independently of each other.
- Method can take advantage of periodicity.
## Pros
- Quadrature for the coarse system becomes easier, i.e.,
  \[ a_h(\lambda_{T,j} - \phi^L_{T,j}, \lambda_{T,j}) \]
- Sparser coarse system
- Less memory consumption, after being computed the correctors \( \phi^L_{T,j} \) can be discarded.

## Cons
- Non-symmetric coarse system
- Harder (missing) analysis
Adaptivity and a posteriori error bound ($b = 0$)

**Theorem (A posteriori error bound)**

Let $u_{H}^{ms,L}$ be the multiscale solution, then

\[
\|\|u - u_{H}^{ms,L}\|\| \lesssim \left(\sum_{T \in \mathcal{T}_H} \rho_{h,T}^2(u_{H}^{ms,L})\right)^{1/2} + \left(\sum_{T \in \mathcal{T}_H} \rho_{L,\omega_T}^2(u_{H}^{ms,L})\right)^{1/2}.
\]

- $\rho_{L,\omega_T}^2$ measures the effect of the truncated patches.
- $\rho_{h,T}^2$ measures the effect of the refinement level.
Adaptivity

- We consider the permeabilities

Figure: Permeabilities One left and SPE right.
• Using a refinement level of 30% we have.

Figure: Convergence plot for One left and SPE right.
Figure: One (left) and SPE (right). The level of refinement (upper) and size of the patches (lower).
Perspective towards Two-Phase flow

Buckley-Leverett system

\[-\nabla \cdot (K \lambda(S) \nabla p) = q \text{ and } \partial_t S + \nabla \cdot (f(s)v) = q_w\]

is solved using IM(plicit)P(ressure)E(plicit)S(aturation)

- $K$ is the hydraulic conductivity
- $\lambda(S)$ is the total mobility (essentially macroscopic)
- and $v = -K\lambda(S)\nabla p$ is obtained from the pressure equation
• Coarse mesh $H = 2^{-5}$ and fine mesh $h = 2^{-8}$.
• Boundary condition $p = 1$, on left boundary $p = 0$ on right boundary, and $K\lambda(S) \nabla p = 0$ otherwise.
• Prepossessing step: compute the basis corrected basis using $\lambda(S) = 1$

Figure: $K_1$ ($A_{max}/A_{min} \approx 5 \cdot 10^5$) left and $K_2$ ($A_{max}/A_{min} \approx 4 \cdot 10^5$) right on a mesh with size $2^{-6}$. 
Introduction and model problem
DG Local Orthogonal Decomposition (DG-LOD)
Petrov-Galerkin DG-LOD
On going work - LOD on complex geometries

Petrov-Galerkin DG-LOD method
Adaptivity
Perspective towards Two-Phase flow

Figure: Saturation profile $K_1$ for $T_1$, $T_2$, and $T_3$.

Figure: Saturation profile $K_2$ for $T_1$, $T_2$, and $T_3$.

<table>
<thead>
<tr>
<th>Data</th>
<th>$|e(T_1)|_{L^2(\Omega)}$</th>
<th>$|e(T_2)|_{L^2(\Omega)}$</th>
<th>$|e(T_3)|_{L^2(\Omega)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.088</td>
<td>0.073</td>
<td>0.070</td>
</tr>
<tr>
<td>2</td>
<td>0.058</td>
<td>0.087</td>
<td>0.079</td>
</tr>
</tbody>
</table>

Table: Error in relative $L^2$-norm, $e(T) = S(T) - S^{\text{ref}}(T)$. 
On going work - LOD on complex geometries

• Construct a method which with textbook convergence which do not resolve the boundary.
• Add correctors locally to handle e.g. singularities and/or interfaces.
Preliminary numerical results

- Homogeneous Dirichlet boundary condition
- Choose $L = \lceil \log(\frac{1}{H}) \rceil$.
- Let $H = \sqrt{2} \cdot 2^{-m}$ for $m = \{2, 3, 4, 5\}$
- Reference mesh is $h = \sqrt{2} \cdot 2^{-8}$
- Holes has radius $r = \{0.01, 0.03\}$ ($\{2^{-6.6439}, 2^{-5.0589}\}$)
- $f = \cos(8\pi x) \cos(8\pi y) + 0.5$

