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Discontinuous Galerkin multiscale methods for convection dominated problems

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Outline

Model problem

Consider the convection-diffusion-reaction problem

$$\begin{aligned} -\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega, \\ u = 0 \text{ on } \partial\Omega. \end{aligned}$$

where $0 < A_{min} \in \mathbb{R} \leq A(x) \in L^\infty(\Omega, \mathbb{R}^{d\times d}_{sym})$, $\mathbf{b} \in [W^1_\infty(\Omega)]^d$, $c \in L^\infty(\Omega)$, $f \in L^2(\Omega)$, with the standard assumption

$$c_o^2 = c - \frac{1}{2}\nabla \cdot \mathbf{b} \geq c_0 \in \mathbb{R} > 0.$$

Discontinuous Galerkin discretization

- Split Ω into elements $\mathcal{T} = \{T\}$, and let $\mathcal{E} = \{e\}$ be the set of all edges in $\mathcal{T}$.

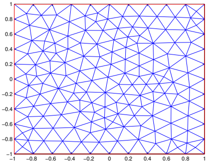

Figure: Example of a mesh on a unit square.

- Let $\mathcal{V}_H$ be the space of all discontinuous piecewise (bi)linear polynomials.

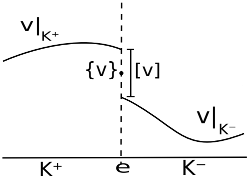

Figure: Example of $\{v\}$ and $[v]$

Discontinuous Galerkin discretization

- $a_H(\cdot,\cdot)$: the diffusion part is approximated using a symmetric interior penalty scheme and the convection part is approximated using upwind.
- The energy-norm is defined by

$$|||\cdot|||_H^2 = ||A^{1/2}\nabla_H \cdot||_{L^2(\Omega)}^2 + \|c_o v\|_{L^2(\Omega)}^2 + \sum_{e\in\mathcal{E}} (\frac{\sigma}{H} + \frac{|\mathbf{b}\cdot\nu|}{2})\|[\cdot]\|_{L^2(e)}^2$$

- Let $\mathcal{V}_H$ be the space of discontinuous piecewise (bi)linear polynomials.

(One scale) DG method

Find $u_H \in \mathcal{V}_H$ such that

$$a_H(u_H, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H.$$

(One scale) DG method for a problem with no convection

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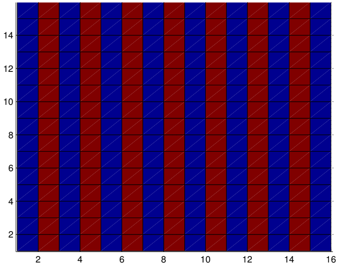

Figure: The coefficient A in the model problem.

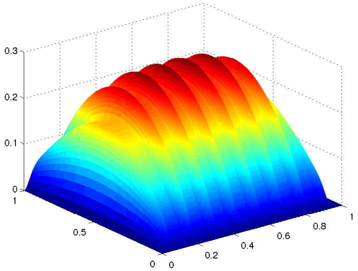

Figure: Reference solution.

(One scale) DG method for a problem with no convection

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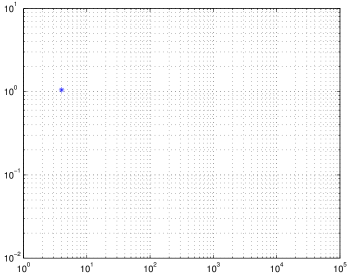

Figure: Energy norm with respect to the degrees of freedom.

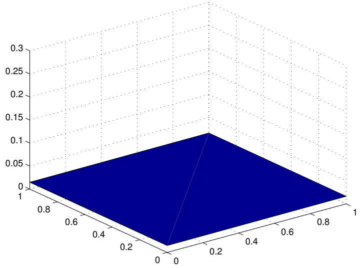

Figure: Solution obtained using the discontinuous Galerkin method.

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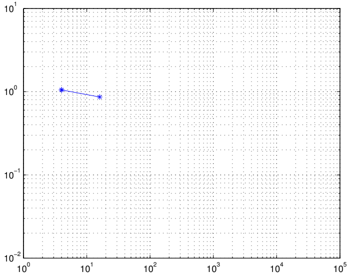

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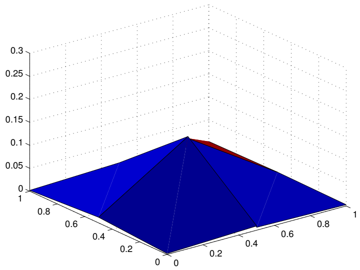

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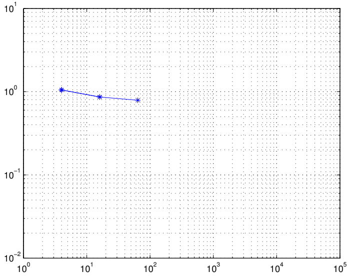

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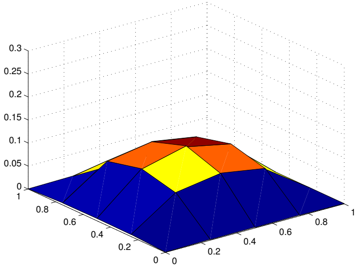

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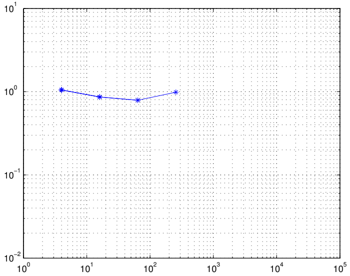

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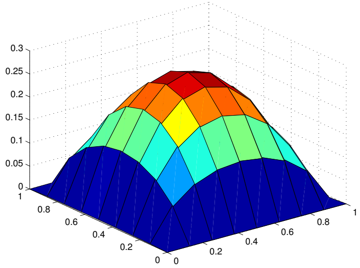

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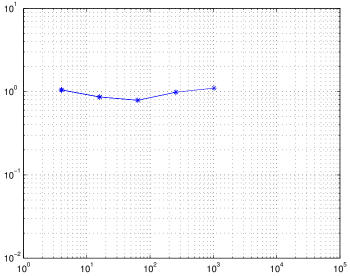

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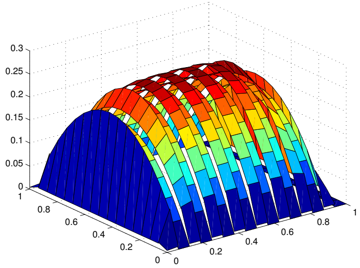

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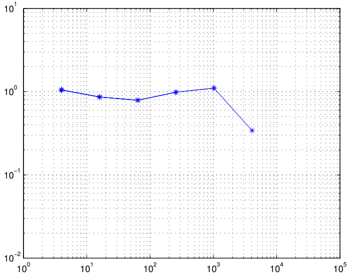

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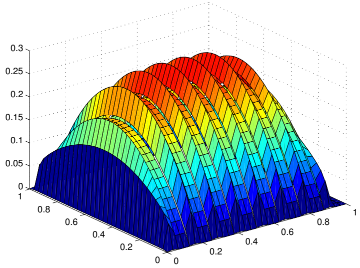

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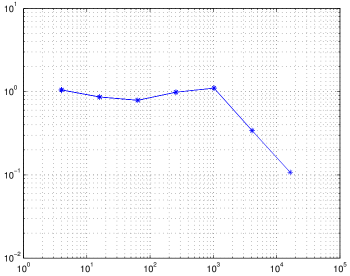

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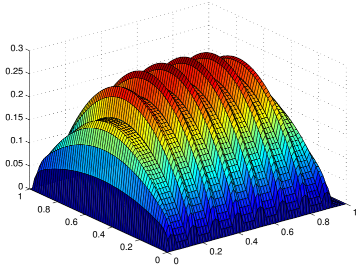

Figure: Solution obtained using the discontinuous Galerkin method.

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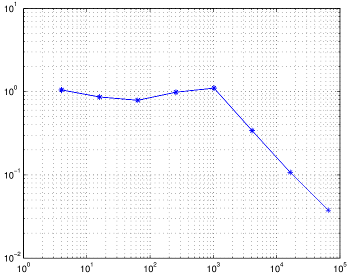

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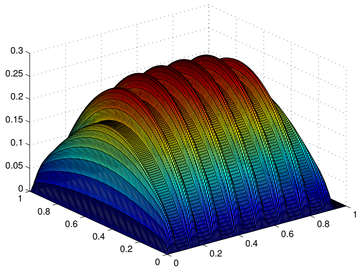


Figure: Solution obtained using the discontinuous Galerkin method.

Objective with the multiscale method

- Eliminate the dependency of A via a multiscale method, i.e.,

$$|||u - u_H^{ms,L}||| \leq C_f H,$$

  where H does not resolve the variation in A.
- Construct an adaptive algorithm to focus the computational effort to critical areas (for the case with pure diffusion).

Some known methods

- Upscaling techniques: Durlofsky et al. 98, Nielsen et al. 98.
- Variational multiscale method: Hughes et al. 95, Arbogast 04, Larson, Målqvist 05, Nolen et al. 08, Nordbotten 09.
- Multiscale FEM: Hou, Wu 96, Efendiev, Ginting 04, Aarnes, Lie 06.
- Residual free bubbles: Brezzi et al. 98.
- Heterogeneous multiscale method: Engquist, E 03, E, Ming, Zang 04, Ohlberger 05.
- Equation free: Kevrekidis et al. 05.
- Metric based upscaling: Owhadi, Zang et al. 06.
- Generlized multiscale FEM: Efendiev, Galvis, Hou 13.
- ...

Remarks

- Local approximations (in parallel) on a fine scale are used to modify a coarse scale space or equation.

Multiscale splitting

- Consider a coarse $\mathcal{V}_H$ and a fine space $\mathcal{V}_h$, such that $\mathcal{V}_H \subset \mathcal{V}_h$.
- Let Π_H be the L^2-projection onto $\mathcal{V}_H$. This will be used as the split between the coarse and fine scale.
- Define $\mathcal{V}^f(\omega) = \{v \in \mathcal{V}_h(\omega) : \Pi_H v = 0\}$.
- We have a L^2-orthogonal split; $\mathcal{V}_h = \mathcal{V}_H \oplus \mathcal{V}^f$.

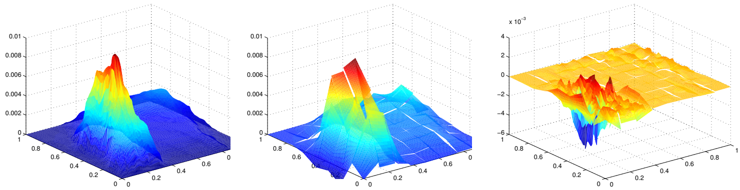


Figure: $u_h = u_H + u^f$

Corrected basis functions

- For each basis function $\lambda_{T,j} \in \mathcal{V}_H$ we calculate a corrector, find $\phi^L_{T,j} \in \mathcal{V}^f(\omega^L_T)$ such that

$$a_h(\phi^L_{T,j}, v_f) = a_h(\lambda_{T,j}, v_f), \quad \text{for all } v_f \in \mathcal{V}^f(\omega^L_T).$$

  where $\text{supp}(\lambda_{T,j}) = T$ and L indicates the size of the patch.
- Let the new corrected space be defined by $\mathcal{V}^{ms}_H = \text{span}\{\lambda_{T,j} - \phi^L_{T,j}\}$.
- We have $u_h = u^{ms}_H + u^f$ where $u_h \in \mathcal{V}_h$, $u^{ms}_H \in \mathcal{V}^{ms}_H$, and $u^f \in \mathcal{V}^f$.

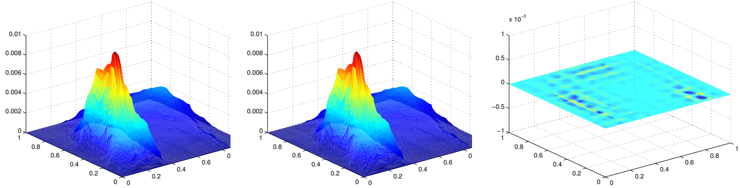
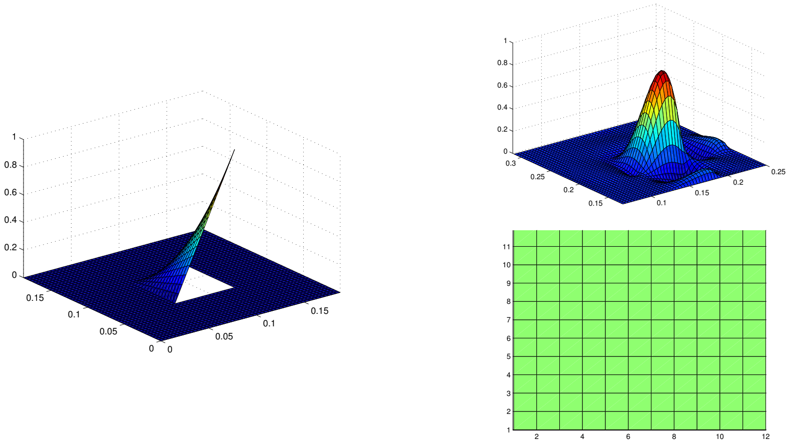


Figure: $u_h = u^{ms}_H + u^f$

Examples of corrected basis functions



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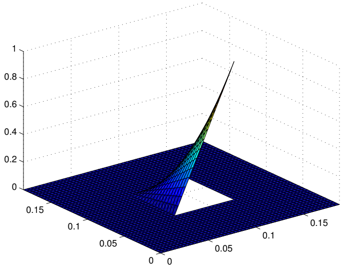
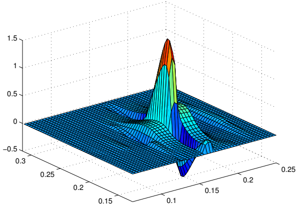
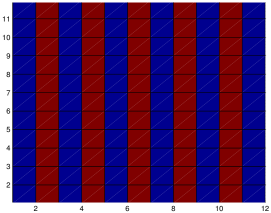
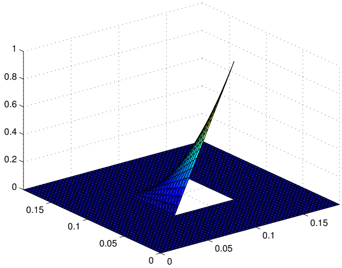
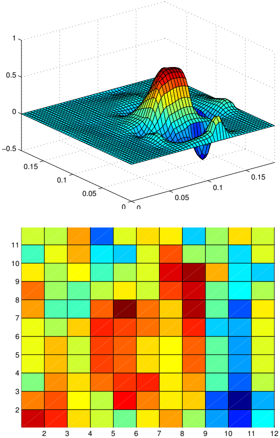






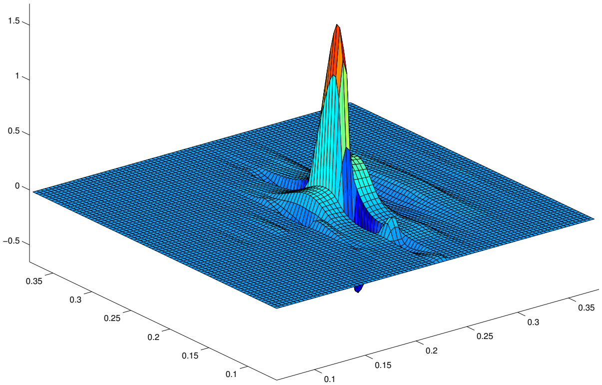
Examples of corrected basis functions





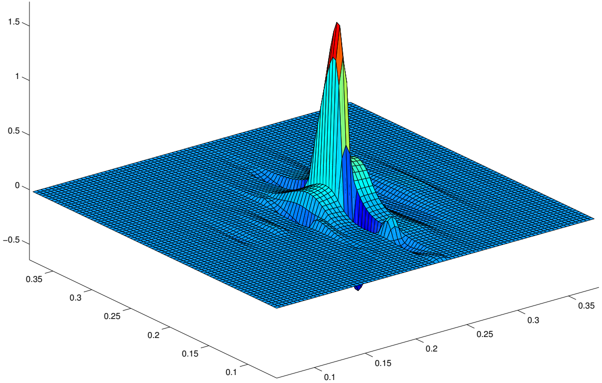
Example of corrected basis functions

- With $\mathbf{b} = [0, 0]'$.



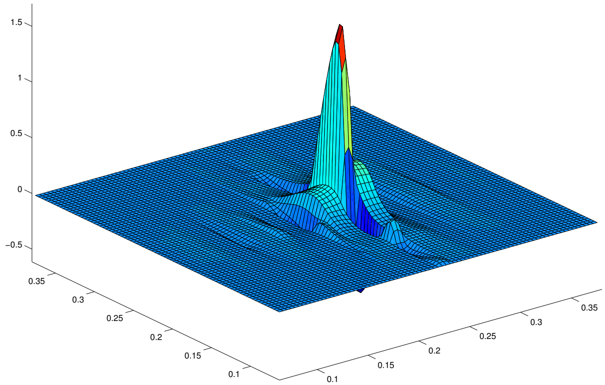
Example of corrected basis functions

- With $\mathbf{b} = -[1, 0]'$.



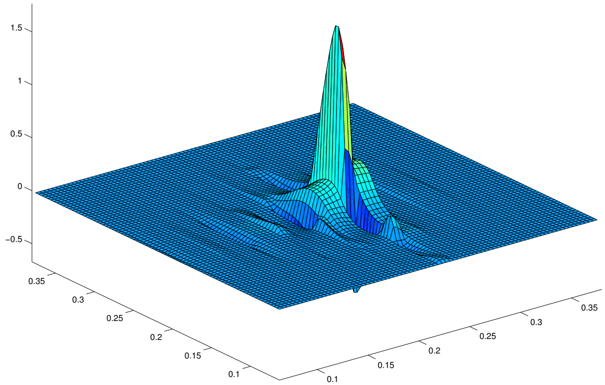
Example of corrected basis functions

- With $\mathbf{b} = -[2, 0]'$.



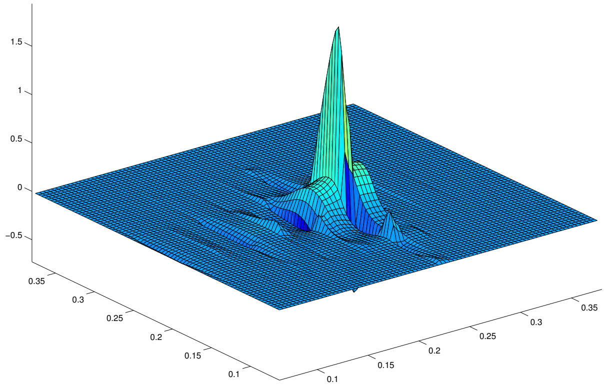
Example of corrected basis functions

- With $\mathbf{b} = -[4, 0]'$.



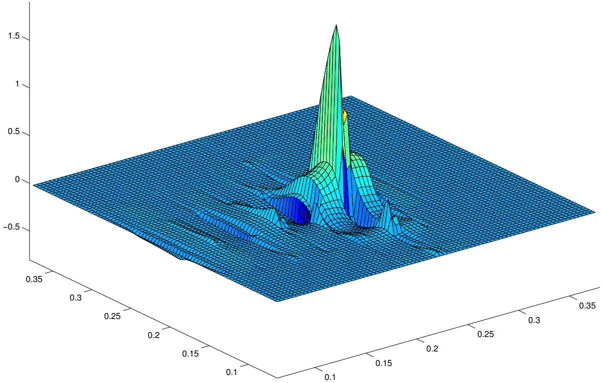
Example of corrected basis functions

- With $\mathbf{b} = -[8, 0]'$.



Example of corrected basis functions

- With $\mathbf{b} = -[16, 0]'$.



Discontinuous Galerkin multiscale method

Consider the problem: find $u_H^{ms,L} \in \mathcal{V}_L^{ms} = \text{span}\{\lambda_{T,j} - \phi_{T,j}^L\}$ such that

$$a_h(u_H^{ms,L}, v) = F(v), \quad \text{for all } v \in \mathcal{V}_H^{ms,L}.$$

- $\dim \mathcal{V}_H^{ms,L} = \dim \mathcal{V}_H$
- The basis functions are computed independently of each other.
- The method can take advantage of periodicity.

Convergence results

ELFVERSON, GEORGOULIS, MÅLQVIST AND PETERSEIM
Convergence of discontinuous Galerkin multiscale methods. *Submitted.*

ELFVERSON AND MÅLQVIST
Discontinuous Galerkin multiscale method for convection dominated problems. *Technical report.*

A priori error bound

Under the assumption $\mathcal{O}(\|A\|_{L^\infty(\Omega)}) = \mathcal{O}(\|H\mathbf{b}\|_{L^\infty(\Omega)})$ the following holds:

Lemma (Decay of corrected basis functions)

For $\phi_{T,j} \in \mathcal{V}^f(\omega_i^L)$, there exists a, $0 < \gamma < 1$, such that

$$|||\phi_{T,j} - \phi_{T,j}^L||| \lesssim \gamma^L |||\lambda_j - \phi_{T,j}|||.$$

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$, there exists a, $0 < \gamma < 1$, such that

$$|||u - u_H^{ms,L}||| \lesssim |||u - u_h||| + ||H(f - \Pi_H f)||_{L^2} + H^{-1}(L)^{d/2}\gamma^L||f||_{L^2}.$$

Choosing $L = \lceil C \log(H^{-1}) \rceil$ both terms behave in the same manner with an appropriate C.

Note: Theorem holds without any assumptions on scales or regularity!

Theorem

For $u_H^{ms,L} \in \mathcal{V}_H^{ms,L}$, such that

$$|||u - u_H^{ms,L}|||_h \leq |||u - u_H|||_h + C_{A_{max}/A_{min},f} H$$

given $f \in L^2(\Omega)$ and choosing $L = \lceil C \log(H^{-1}) \rceil$.

Note: Theorem holds without any assumptions on scales or regularity!

Numerical verification of the convergence

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

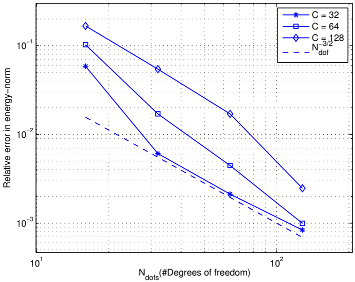


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Let $A = 1$, $c = 0$, and $\mathbf{b} = C[1, 0]'$ for $C = 32, 54, 128$.
- Choose $L = \lceil 2\log(\frac{1}{H})\rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{2, 3, 4, 5\}$.
- Reference mesh is 2^{-7}.

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $c = 0$, and $\mathbf{b} = [1, 0]'$.

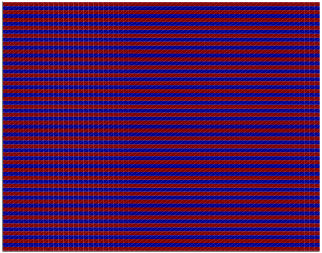


Figure: Diffusion coefficient A, $A_{max}/A_{min} = 100$ and $A_{min} = 0.01$.

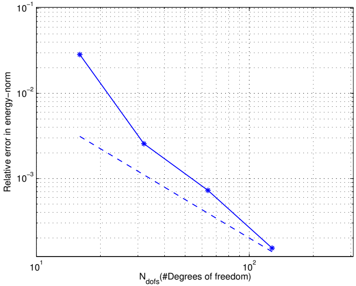


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

$$-\nabla \cdot A\nabla u + \mathbf{b} \cdot \nabla u + cu = f \text{ in } \Omega,$$
$$u = 0 \text{ on } \partial\Omega.$$

- Let $c = 0$, and $\mathbf{b} = [512, 0]'$.

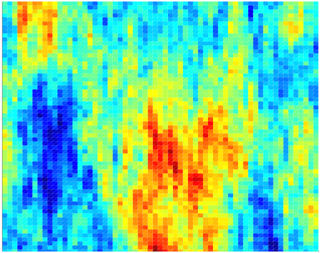


Figure: Diffusion coefficient A with $A_{max}/A_{min} \sim 10^5$.

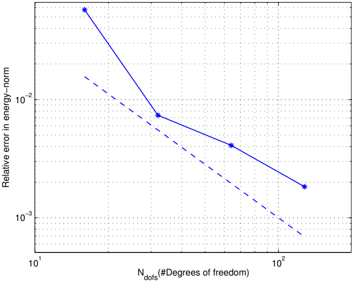


Figure: $\#dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

Pure diffusion on L-shaped domain

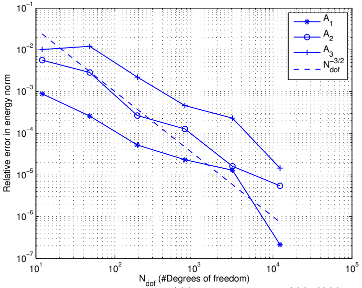


Figure: $\# dofs$ vs $|||u_h - u^{ms}_{H,L}|||/|||u_h|||$

- Choose $L = \lceil 2\log(\frac{1}{H}) \rceil$.
- Let the right hand side be: $f = 1 + sin(\pi x) + sin(\pi y)$.
- Let $H = 2^{-m}$ for $m = \{1, 2, 3, 4, 5, 6\}$.
- Reference mesh is 2^{-8}.

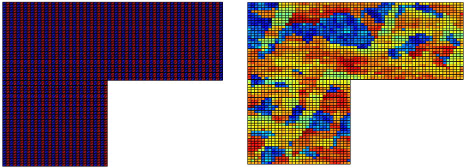


Figure: Permeabilities are piecewise constant on a mesh with size 2^{-5}, with ratio $A_{max}/A_{min} = \{10, 7 \cdot 10^6\}$

Adaptivity and a posteriori error bound for pure diffusion

Theorem (A posteriori error bound)

Let $u_H^{ms,L}$ be the multiscale solution, then

$$|||u - u_H^{ms,L}||| \lesssim \left(\sum_{T \in \mathcal{T}_H} \rho_{h,T}^2 \right)^{1/2} + \left(\sum_{T \in \mathcal{T}_H} \rho_{L,\omega_T^L}^2 \right)^{1/2}.$$

- $\rho_{L,\omega_i^L}^2$ and $\rho_{h,K}^2$ depends on $u_H^{ms,L}$
- $\rho_{L,\omega_i^L}^2$ measures the effect of the truncated patches.
- $\rho_{h,T}^2$ measures the effect of the refinement level.

Elfverson, Georgoulis, and Målqvist
An adaptive discontinuous Galerkin multiscale method for elliptic problems. *SIAM Multiscale Modeling and Simulations (MMS).*

Numerical experiment

- We consider the permeabilities

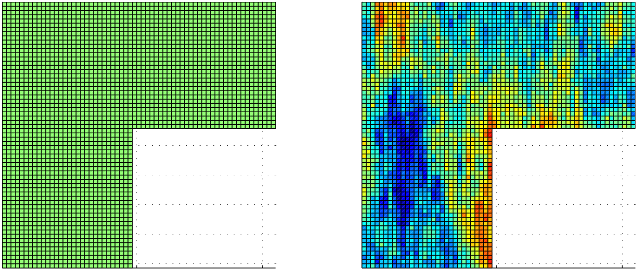


Figure: Permeabilities *One* left and *SPE* right.

Numerical experiments

- We use a refinement level of 30%.

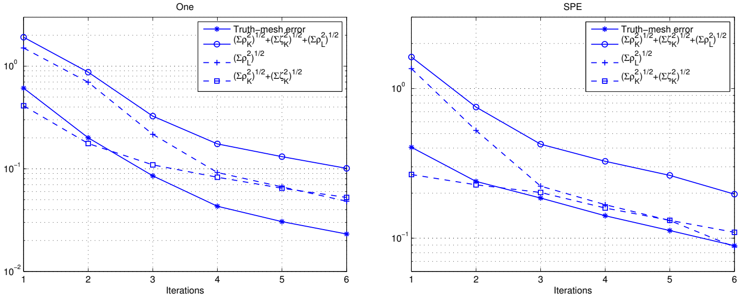


Figure: Convergence plots for *One* left and *SPE* right.

Numerical experiments

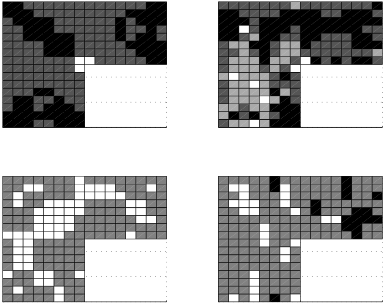


Figure: The level of refinement and size of the patches illustrated in the upper resp. lower plots for the different permeability One (left) and SPE (right). White is where most refinement resp. larger patch are used and black is where least refinement resp. smallest patches are used.

The End