Adaptive discontinuous Galerkin multiscale methods for elliptic problems

Energy norm a posteriori error estimate

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Model problem

Poisson’s equation

Given a polygonal domain $\Omega \subset \mathbb{R}^d$. We want to find $u$ such that

$$-\nabla \cdot \alpha \nabla u = f \quad \text{in } \Omega,$$

$$n \cdot \nabla u = 0 \quad \text{on } \partial \Omega,$$

where $\alpha$ is bounded $0 < \beta \leq \alpha(x) \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $\int_\Omega f \, dx = 0$.

(a) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$

(b) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$

(c) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$

(d) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^6$

Figure: Permeabilities $\alpha$ projected in log scale and taken from the Society of Petroleum Engineer http://www.spe.org/
Discontinuous Galerkin discretization

Discretization

- Let \( \Omega \) be subdivided into the partition \( \mathcal{K} = \{K\} \) and \( \Gamma^I \) be the union of all interior edges.
- Let also \( \mathcal{V}_h \) be the space of all discontinuous piecewise (bi)linear polynomials.
- Define the weighted average and jump on face \( e \) as:

\[
\{v\}_w = \frac{\alpha^+ v^-}{\alpha^+ + \alpha^-} + \frac{\alpha^- v^+}{\alpha^+ + \alpha^-} \quad \text{and} \quad [v] = v^+ - v^-.
\]

(a) Here \( \mathcal{K} = \{K^+, K^-\} \) and \( \Gamma^I = \{e\} \)

(b) Example of \( \{v\} \) and \( [v] \)
Consider a symmetric inconsistent interior penalty discontinuous Galerkin method

- Expanded DG space: \( \mathcal{V} = \mathcal{V}_h + H^{1+\epsilon} \) with \( \epsilon > 0 \).
- Denote \( \Pi : (L^2(\Omega))^d \rightarrow (\mathcal{V}_h)^d \) the \( L^2 \)-projection onto \( (\mathcal{V}_h)^d \)

The bilinear form \( a(\cdot, \cdot) : \mathcal{V} \times \mathcal{V} \rightarrow \mathbb{R} \) and right hand side \( l(\cdot) : \mathcal{V} \rightarrow \mathbb{R} \) are defined as:

\[
a(v, z) = \sum_{K \in \mathcal{K}} (\alpha \nabla v, \nabla z)_{L^2(K)} - \sum_{e \in \Gamma^I} \left( (n \cdot \{\alpha \Pi \nabla v\}_w, [z])_{L^2(e)} + (n \cdot \{\alpha \Pi \nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e \gamma_e}{h_e} ([v], [z])_{L^2(e)} \right),
\]

\[
l(v) = (f, v)_{L^2(\Omega)}.\]
Comments

- Why use weighted averages?
- Using $\Pi$, since we want to assume as little regularity as possible in $u$ for the a posteriori error analysis.
- For $\nu \in (\mathcal{V}_h)^d$ then $\Pi \nu = \nu$ and $a(\cdot, \cdot)$ is reduced to a more familiar fashion.
Motivation
In many applications, solution exist on several different scales e.g. flow in porous media and in composite materials.
- Secondary oil recovery.
- Sequestration of Carbon Dioxide.
Why do we need to resolve the coefficients?

Example with periodic coefficient

Consider Possion’s equation with period coefficient $\alpha = \alpha(x/\epsilon)$. For the finite element method, we have

$$\|\sqrt{\alpha} \nabla (u - u_h)\|_{L^2(\Omega)} \leq C \frac{H}{\epsilon} \|f\|_{L^2(\Omega)}$$

- Need $H \ll \epsilon$ for reliable results.
- Too computational expensive to solve on a single mesh for many applications e.g. flow in porous media and in composite materials.
- Want eliminate the $\epsilon$ dependence by using a multiscale method (Målqvist-Peterseim).
Framework for Multiscale methods

The problem is split into one coarse and fine scale contribution
\[ \mathcal{V}_h = \mathcal{V}_c \oplus \mathcal{V}_f. \]

- Let subdivide \( \Omega \) into a coarse mesh \( \mathcal{K}_c = \{ K_c \} \).
- \( \mathcal{V}_c = \text{span}\{ \phi_i \} = \Pi_c \mathcal{V}_h \) and \( \mathcal{V}_f = \{ v \in \mathcal{V}_h : \Pi_c v = 0 \} \), where \( \Pi_c : \mathcal{V}_h \to \mathcal{V}_c \) is the \( L^2 \) projection onto the coarse mesh.
- Define the map \( \mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f \) as
  \[ a(\mathcal{T} v_c, v_f) = -a(v_c, v_f), \quad \forall v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f \]

Split \( u_h = u_c + \mathcal{T} u_c + u_f \) and \( v = v_c + v_f \) where \( u_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f \).

\[ a(u_c + \mathcal{T} u_c + u_f, v_c + v_f) = l(v_c + v_f), \quad \forall v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f \]

\( u_h = u_c + \mathcal{T} u_c + u_f \)
**Fine scale**
Let $v_c = 0$ to get the fine scale equations

$$a(Tu_c + uf, v_f) = l(v_f) - a(u_c, v_f),$$

split into two equations

$$a(u_f, v_f) = l(v_f) \quad \forall v_f \in \mathcal{V}_f,$$

$$a(Tu_c, v_f) = -a(u_c, v_f) \quad \forall v_f \in \mathcal{V}_f.$$

**Coarse scale**
Let $v_f = 0$ on the coarse scale

$$a(u_c + Tu_c, v_c) = l(v_c) - a(u_f, v_c) \quad \forall v_c \in \mathcal{V}_c$$

**Comments**

- Equally hard to solve as the original problem.
- Other chooses then $\Pi_c$ can be coincided.
- A symmetric split can also be considered for the coarse scale problem.
View solution as span av modified basis functions

- Let $V_c = \text{span}\{\phi_i\}$ and $V^{ms} = \text{span}\{\phi_i + T\phi_i\}$.
- View $\phi_i + T\phi_i$ as a modified basis function.

From the multiscale map we have, $V_h = V^{ms} \perp_a V_f$, for all $i$

$$a(\phi_i + T\phi_i, v) = 0, \quad v \in V_f$$
Approximation of $\mathcal{T} \phi_i$

- The fast decay of $\mathcal{T} \phi_i$ motivates approximations of $\mathcal{T} \phi_i$ to patches $\omega_i^L \subset \Omega$.

**Figure:** Example of a one layer patch $\omega_i^1$ and a two layer patch $\omega_i^2$. 
Multiscale method discretization

- $\tilde{T}$ is the restriction of $T$ to a patch $\omega \subset \Omega$
- $\tilde{U}_f = \sum_{i \in \mathcal{N}} \tilde{U}_{f,i}$ where $\mathcal{N}$ is the number of nodes, be the approximation of $u_f$.
- Let $\mathcal{M}_i$ be all $j$ s.t. $\phi_j = 1$ in node $i$.
- Let also $\Phi_i = \sum_{j \in \mathcal{M}_i} \phi_j$

![Graphs showing $\phi_j$ and $\Phi_i$](image_url)
Bilinear form for the fine scale problem

- Let $\mathcal{K}(\omega_L^i) = \{K : K \cap \omega_L^i \neq 0\}$.
- Let also $\Gamma^I(\omega_L^i)$ be all interior edges on $\mathcal{K}(\omega_L^i)$.

Define $a_i : \mathcal{V}_f(\omega_i) \times \mathcal{V}_f(\omega_i) \rightarrow \mathbb{R}$, as

$$a_i(v, z) = \sum_{K \in \mathcal{K}(\omega_L^i)} (\alpha \nabla v, \nabla z)_{L^2(K)} - \sum_{e \in \Gamma^I(\omega_L^i)} \left( (\mathbf{n} \cdot \{\alpha \nabla v\}_w, [z])_{L^2(e)} + (\mathbf{n} \cdot \{\alpha \nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma e \gamma_e}{h_e} ([v], [z])_{L^2(e)} \right),$$

$$l_i(v) = (\Phi_i f, v)_{L^2(\Omega)}.$$
Fine scale equations
For all $i \in \mathcal{N}$: find $\tilde{T}\phi_j \in \mathcal{V}_f(\omega_i^L)$ and $U_{f,i} \in \mathcal{V}_f(\omega_i^L)$ for $j \in \mathcal{M}_i$ s.t

$$a_i(\tilde{T}\phi_j, v_f) = -a_i(\phi_j, v_f), \quad \forall v_f \in \mathcal{V}_f(\omega_i^L),$$

$$a_i(\tilde{U}_{f,i}, v_f) = l_i(\Phi_i v_f), \quad \forall v_f \in \mathcal{V}_f(\omega_i^L).$$

Coarse scale equation
Find $U_c \in \mathcal{V}_c$ s.t

$$a(U_c + \tilde{T}U_c, v_c) = l(v_c) - (\tilde{U}_f, v_c), \quad \forall v_c \in \mathcal{V}_c.$$
Decay in $V_f$

Problem setting

- Let the computational domain be $\omega^L_i$ for $L = 1, 2, \ldots, N$ where $\omega^L_i \subseteq \Omega$.
- Let also $\Phi_i = \sum_{j \in M_i} \phi_j$
- The problem reads: find $\tilde{T}\Phi_i \in V_h(\omega^L_i)$

$$a(\tilde{T}\Phi_i, v) = -a(\Phi_i, v), \quad \forall v \in V_h(\omega^L_i).$$

- The reference solution $T\Phi_i$ is the solution computed on $\omega^N_i = \Omega$.
- Coarse mesh is $8 \times 8$ element and reference grid is $64 \times 64$ elements.
Figure: The error in relative error in broken energy norm with respect to the path size.
Convergence

Problem setting

- Consider the model problem (Poisson’s equation)
- Keeping the refinement level constant and increasing the patch sizes $L = 1, \ldots, N$ for all local problems.
- The coarse grid is $8 \times 8$ coarse elements.
- The reference solution $U_{ref}$ is the DG solution computed on $64 \times 64$ elements.
- The right hand side is $-1$ in the lower left corner and $1$ in the upper right.
Figure: Permeabilities $\alpha$.

Figure: The reference solution to the model problem using the permeabilities *One, Period* and *SPE*.
Figure: The relative error in broken energy norm with respect to the patch sizes.
Implementation

Split into local problems, transfer data

\[ a, f, \Omega \]

Solve independent local fine scale equations

\[ T\phi_j \ \forall j \in M_1 \]
\[ U_{f,1} \]
\[ K_{(j),b_1} \]

\[ T\phi_j \ \forall j \in M_2 \]
\[ U_{f,2} \]
\[ K_{(j),b_2} \]

\[ T\phi_j \ \forall j \in M_3 \]
\[ U_{f,3} \]
\[ K_{(j),b_3} \]

\[ \cdots \]

Generate and solve coarse scale equation

\[ KU_c = \sum b_i \]

Generate the solution of the multiscale equation \( U \) or interesting parts of \( U \)

\[ U = \sum U_{c,i} (\phi_i + T\phi_i) + U_f \]

Figure: Scheme of the implementation.
Constraints on the fine scale equations

- The condition is realised using Lagrangian multiplier.
- Let $\phi$ be a coarse basis function and $\varphi$ be a fine basis function.

Want so fined $\tilde{T}w \in \mathcal{V}_f(\omega_i^L)$

$$a_i(\tilde{T}w, v) = -a_i(w, v) \quad \forall v \in \mathcal{V}_f(\omega_i^L).$$

Algebraic problem reads:

$$
\begin{pmatrix}
K & P^T \\
P & 0
\end{pmatrix}
\begin{pmatrix}
\xi \\
b
\end{pmatrix}
=
\begin{pmatrix}
b_0 \\
0
\end{pmatrix},
$$

where $K_k,l = a_i(\varphi_k, \varphi_l)$, $b_k = -a_i(\phi_l, \varphi_k)$ and

$$P = \begin{pmatrix}
(\phi_1, \varphi_1) & (\phi_1, \varphi_2) & \cdots & (\phi_1, \varphi_N) \\
(\phi_2, \varphi_1) & (\phi_2, \varphi_2) & \cdots & (\phi_2, \varphi_N) \\
\vdots & \vdots & \ddots & \vdots \\
(\phi_M, \varphi_1) & (\phi_M, \varphi_2) & \cdots & (\phi_M, \varphi_N)
\end{pmatrix}.$$
Adaptivity

Set up

- Need an a posteriori error estimate for the discontinuous Galerkin method.
- Use this in the framework for Multiscale methods to construct a a posteriori error estimate for the multiscale method.
- Construct a adaptive algorithm to automatically tune the critical parameters.
Theorem (A posteriori error estimate for DG method)

- Let $u$, $u_h$ be given by the exact solution respectively the DG solution.
- Let also $\chi \in \mathcal{V}_h \cap H^1(\Omega)$
- Moreover, let $\mathcal{E} := \mathcal{E}_c + \mathcal{E}_d$ where $\mathcal{E}_c := u - \chi$ and $\mathcal{E}_d := \chi - u_h$.

Then,

$$
\sum_{K \in \mathcal{K}} \left\| \sqrt{\alpha} \nabla \mathcal{E}_c \right\|_{L^2(K)}^2 \lesssim \sum_{K \in \mathcal{K}} \left( \rho_K(u_h) + \zeta_K(u_h, \chi) \right)^2,
$$

where

$$
\rho_K(u_h) = \frac{h_K}{\sqrt{\alpha_0}} \left\| f + \nabla \cdot \alpha \nabla u_h \right\|_{L^2(K)},
$$

$$
+ \sqrt{\frac{h_K}{\alpha_0}} \left( \left\| (1 - w_K(e)) n \cdot [\alpha \nabla u_h] \right\|_{L^2(\partial K)} + \left\| \frac{\sigma e \gamma e}{h_e} [u_h] \right\|_{L^2(\partial K \setminus \Gamma^B)} \right),
$$

$$
\zeta_K(u_h, \chi) = \left\| \sqrt{\alpha} \nabla (u_h - \chi) \right\|_{L^2(K)}.
$$
Treatment of $\zeta_K(u_h, \chi)$

1. First, $\chi$ has to be chosen in a clever way.
2. Second, $\zeta_K(u_h, \chi)$ can either be estimated or evaluated.

- One possible chose is $\chi = I_{Os}u_h$.
- Under certain assumption on $\alpha$, $\zeta_K(u_h, \chi)$ can be evaluated and hidden in $\varrho_K(u_h)$.

Lemma (Oswald interpolation operator)

- Let $I_{Os} : \mathcal{V}_h \to \mathcal{V}_h \cap H^1$
  \[ I_{Os}v = \sum_{i \in N} \left( \frac{1}{|M_j|} \sum_{j \in M_i} v_j(x_i) \varphi_j \right). \]

Then,
\[ \|\sqrt{\alpha} \nabla (v - I_{Os}v)\|^2 \lesssim \alpha^0 \|\frac{1}{\sqrt{h_e}} [v]\|^2_{L^2(\partial K \setminus \Gamma^B)}. \]
Sketch of proof

We have

\[ \sum_{K \in \mathcal{K}} \| \sqrt{\alpha} \nabla \mathcal{E}_c \|^2_{L^2(K)} = a(\mathcal{E}_c, \mathcal{E}_c) = a(\mathcal{E}, \mathcal{E}_c) - a(\mathcal{E}_d, \mathcal{E}_c), \]

where

\[ a(\mathcal{E}, \mathcal{E}_c) = a(u, \mathcal{E}_c) - a(u_h, \mathcal{E}_c) = l(\mathcal{E}_c) - a(u_h, \mathcal{E}_c), \]

\[ = l(\eta) - a(u_h, \eta), \]

where \( \eta = \mathcal{E}_c - \pi_0 \mathcal{E}_c. \)

- First integration by parts \( l(\eta) - a(u_h, \eta) \) element wise and using the identity \([vz] = \{v\}_w[z] + \{v\}_{\bar{w}}[z].\)
We get,
\[ l(\eta) - a(u_h, \eta) = \sum_{K \in K} (f + \nabla \cdot \alpha \nabla u_h, \eta)_{L^2(K)} + \sum_{e \in \Gamma^I} \left( - (n \cdot [\alpha \nabla u_h], \{\eta\} \bar{w})_{L^2(e)} \right) + \sum_{e \in \Gamma^B} (n \cdot \{\alpha \Pi \nabla \eta\} \bar{w}, [u_h])_{L^2(e)} - \sigma \gamma e h^{-1}_e ([u_h], [\eta])_{L^2(e)} + \sum_{e \in \Gamma^B} (n \cdot \alpha \nabla u_h, \eta)_{L^2(e)}.\]

1. Then, using the inequalities and stability for the piecewise constant \( L^2 \)-projection.

\[ \|v - \pi_0 v\|_{L^2(K)} \lesssim \frac{h_K}{\sqrt{\alpha_0}} \|\sqrt{\alpha} \nabla v\|_{L^2(K)}, \quad \forall v \in H^1(K), \]

\[ \|v - \pi_0 v\|_{L^2(\partial K)} \lesssim \sqrt{\frac{h_K}{\alpha_0}} \|\sqrt{\alpha} \nabla v\|_{L^2(K)} \quad \forall v \in H^1(K). \]

2. For \( a(\mathcal{E}_d, \mathcal{E}_c) \) use the Lemma (Oswald interpolation operator).
Theorem (A posteriori error estimate for ADG-MS)

- Let \( u, U \) be the exact solution respectively the multiscale solution.
- Let \( X = \mathcal{I}_{Os} U \in H^1(\Omega) \).
- Set \( \mathcal{E} := \mathcal{E}_c + \mathcal{E}_d \) where \( \mathcal{E}_c := u - X \) and \( \mathcal{E}_d := X - U \).
- \( U_i := \sum_{j \in M_i} U_{c,j} (\phi_j + \tilde{T}_j \phi_j) + U_{f,i} \), where \( U_{c,j} \) are the nodal values.

Then,

\[
\sum_{K \in \mathcal{K}} \| \sqrt{\alpha} \nabla \mathcal{E}_c \|_{L^2(K)}^2 \lesssim \sum_{K_c \in \mathcal{K}_c} \rho_{h,K_c}^2 + \sum_{i \in \mathcal{N}} \rho_{L,\omega_i^L}^2,
\]

where

\[
\rho_{L,\omega_i^L}^2 = \sum_{e \in \Gamma^B(\omega_i^L) \setminus \Gamma^B} \rho_{L,\omega_i^L,e}^2,
\]

\[
\rho_{L,\omega_i^L,e} = \frac{H_{\omega_i^L}}{\sqrt{h_K \alpha_0}} \left( \| n \cdot \{ \alpha \nabla U_i \}_w \|_{L^2(e)} + \frac{\sigma_e \gamma_e}{h_e} \|[U_i]\|_{L^2(e)} \right),
\]

measures the effect of the truncated patches.
Also

\[
\rho_{h,K_c}^2 = \sum_{K \in K_c} (\varrho_K(U) + \zeta_K(U, X))^2,
\]

with \( \varrho_K \) and \( \zeta_K \) as in previous theorem.

**Comments**

- \( \rho_{L,\omega_i}^2 \) measure the effect of the truncated patches.
- \( \rho_{h,K}^2 \) measure the effect of the refinement level.
- \( \frac{1}{\sqrt{h_K h_e}} \|U_i\|_{L^2(e)} \) behave as \( h^{-3/2} e^{-L} \sim 1 \Rightarrow L \sim \frac{3}{2} \log h^{-1} \)
- Another possible choice is a weighted Oswald-type interpolation operator with the weights depending on the diffusion tensor.
Sketch of proof

We have

\[ a(\mathcal{E}_c, \mathcal{E}_c) = a(\mathcal{E}, \mathcal{E}_c) - a(\mathcal{E}_d, \mathcal{E}_c), \]

where

\[ a(\mathcal{E}, \mathcal{E}_c) = a(u, \mathcal{E}_c) - a(U, \mathcal{E}_c), \]

\[ = l(\mathcal{E}_c) - a(U, \mathcal{E}_c), \]

\[ = l(\mathcal{E}_c - \nu_c) - a(U, \mathcal{E}_c - \nu_c), \]

\[ = \sum_{i \in \mathcal{N}} \left( l_i(\mathcal{E}_c - \nu_c - \nu_f) - a(U_i, \mathcal{E}_c - \nu_c) + a_i(U_i, \nu_f) \right). \]

here \( \nu_c \in \mathcal{V}_c \) and \( \nu_f \in \mathcal{V}_f \).

Notice that

\[ a_i(\mathcal{E}, \mathcal{E}_c) = a(\mathcal{E}, \mathcal{E}_c) + \sum_{e \in \Gamma^B(\omega_i^L)} \left( (n \cdot \{ \alpha \nabla U_i \})_w, [v_f] \right)_{L^2(e)} \]

\[ + (n \cdot \{ \alpha \nabla U_i \})_w, [v_f] \right)_{L^2(e)} - \frac{\sigma_e \gamma_e}{h_e} ([U_i], [v_f])_{L^2(e)} \]
Then,

\[
a(\mathcal{E}, \mathcal{E}_c) = \left( I(\mathcal{E}_c - v_c - v_f) - a(\mathcal{E}, \mathcal{E}_c - v_c - v_f) \right) \\
+ \sum_{i \in N} \sum_{e \in \Gamma^B(\omega_i^f) \Gamma^B} \left( (n \cdot \{\alpha \nabla U_i\}_w, [v_f])_{L^2(e)} + (n \cdot \{\alpha \nabla U_i\}_w, [v_f])_{L^2(e)} \right) \\
- \frac{\sigma e \gamma e}{h_e} ([U_i], [v_f])_{L^2(e)} \right) \\
=: I + II.
\]

1. The first term (I), is bounded by the a apoteriori error estimate for DG.

2. To bound the second term (II),
   - Select \(v_c\) and \(v_f\) as the piecewise constant \(L^2\)-projection onto \(V_c\) and \(V_f\), respectively.
   - Then using a trace inequality, a interpolation estimate and \(L^2\)-stability of \(\pi_f\), II is bounded.
Adaptive algorithm

Algorithm 1 Adaptive Discontinuous Galerkin Multiscale Method

1: Initialize the coarse mesh with mesh size $H$.
2: Let the fine mesh size be $h_K = H/2$ for all $K_c \in \mathcal{K}_c$ and $L(\omega_i) = 2$ for all $i \in \mathcal{N}$
3: while $\sum_{i \in \mathcal{N}} \left( \rho_{h,\omega_i}^2 + \rho_{L,\omega_i}^2 \right) > TOL$ do
4: for $i \in \mathcal{N}$ do
5: if $\rho_{L,\omega_i}^2 > TOL/(2\mathcal{N})$ then
6: $L(\omega_i) := L(\omega_i) + 1$
7: end if
8: end for
9: for $K_c \in \mathcal{K}_c$ do
10: if $\rho_{h,K}^2 > TOL/(2|\mathcal{K}_c|)$ then
11: $h_K := h_K/2$
12: end if
13: end for
14: end while
Adaptivity

- Consider the model problem
- Using the a posteriori error estimate to construct an adaptive algorithm.
- Start with one refinement and 2 layers patches everywhere.
- Refine 30% of the coarse elements and increase 30% of the patch sizes in each iteration.
- Coarse mesh is $32 \times 32$ element and reference grid is $256 \times 256$ elements.
- The right hand side is $-1$ in the lower left corner and $1$ in the upper right.

(a) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$  
(b) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$  
(c) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^5$  
(d) $\frac{\alpha_{\text{max}}}{\alpha_{\text{min}}} \sim 10^6$

**Figure:** Permeabilities $\alpha$ projection in log scale.
Figure: The relative error in broken energy norm with respect to number of iterations. Iteration 0 corresponds to the standard DG solution and iteration 1 the start values in the adaptive algorithm.
Figure (a) and (b) illustrates where the adaptive algorithm puts most effort

- Figure (a) corresponds to the refinements
- Figure (b) corresponds to the patch sizes.
- Figure (c) is the permeability $\alpha$.

(a) Refine $h_K$  (b) Layers, $L$

(c) Layers, $L$
Figure: The relative error in broken energy norm with respect to the mean value of the degrees of freedom for the fine scale problems.
Conclusions

Advantage

- The fine scale problems are perfectly parallelizable.
- The exponential decay in the fine scale solution allows small patches.
- The error estimate and the adaptivity algorithm focus computational effort in critical areas.
- Very high aspect ratio in $\alpha$ can be solved.
- Possible to construct a conservative flux on the coarse scale.
Future work

- A priori analysis of the discontinuous Galerkin multiscale method.
- Hybrid method using DG on coarse scale and CG on fine scale.
- Extend the method and analysis to diffusion-convection problems.
- Investigate to sensibility in input data (uncertainty in the permeability $\alpha$)
- Extend the implementation to triangular meshed to allow for complicated geometries.
- 3D implementation.
Questions