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Adaptive discontinuous Galerkin multiscale methods for elliptic problems

Energy norm a posteriori error estimate

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Outline

Papers

- D. Elfverson, E. Georgoulis and A. Målqvist, Adaptive discontinuous Galerkin multiscale method (submitted).

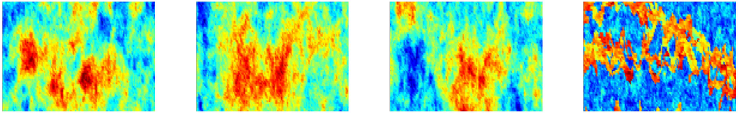
Model problem

Poisson's equation

Given a polygonal domain $\Omega \subset \mathbb{R}^d$. We want to find u such that

$$-\nabla \cdot \alpha \nabla u = f \text{ i } \Omega,$$
$$n \cdot \nabla u = 0 \text{ on } \partial\Omega,$$

where α is bounded $0 < \beta \leq \alpha(x) \in L^\infty(\Omega)$, $f \in L^2(\Omega)$ and $\int_\Omega f\, dx = 0$.



(a) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (b) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (c) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$ (d) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^6$

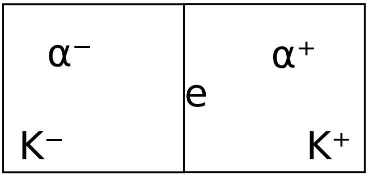
Figure: Permeabilities α projected in log scale and taken from the Society of Petroleum Engineer `http://www.spe.org/`

Discontinuous Galerkin discretization

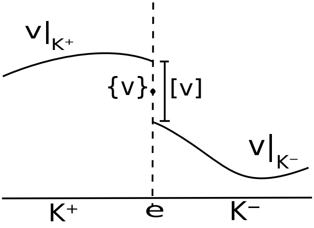
Discretization

- Let Ω be subdivided into the partition $\mathcal{K} = \{K\}$ and Γ^I be the union of all interior edges.
- Let also $\mathcal{V}_h$ be the space of all discontinuous piecewise (bi)linear polynomials.
- Define the weighted average and jump on face e as:

$$\{v\}_w = \frac{\alpha^+ v^-}{\alpha^+ + \alpha^-} + \frac{\alpha^- v^+}{\alpha^+ + \alpha^-} \quad \text{and} \quad [v] = v^+ - v^-.$$



(a) Here $\mathcal{K} = \{K^+, K^-\}$ and $\Gamma^I = \{e\}$



(b) Example of $\{v\}$ and $[v]$

Consider a symmetric inconsistent interior penalty discontinuous Galerkin method

- Expanded DG space: $\mathcal{V} = \mathcal{V}_h + H^{1+\epsilon}$ with $\epsilon > 0$.
- Denote $\Pi : (L^2(\Omega))^d \to (\mathcal{V}_h)^d$ the L^2-projection onto $(\mathcal{V}_h)^d$

The bilinear form $a(\cdot,\cdot) : \mathcal{V} \times \mathcal{V} \to \mathbb{R}$ and right hand side $l(\cdot) : \mathcal{V} \to \mathbb{R}$ are defined as:

$$
\begin{aligned}
a(v,z) = \sum_{K\in\mathcal{K}} (\alpha\nabla v, \nabla z)_{L^2(K)} - \sum_{e\in\Gamma^I} \Big(&(\mathbf{n}\cdot\{\alpha\Pi\nabla v\}_w, [z])_{L^2(e)} \\
&+ (\mathbf{n}\cdot\{\alpha\Pi\nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e\gamma_e}{h_e}([v],[z])_{L^2(e)}\Big), \\
l(v) = (f, v)_{L^2(\Omega)}.&
\end{aligned}
$$

Comments

- Why use weighted averages?
- Using Π, since we want to assume as little regularity as possible in u for the a posteriori error analysis.
- For $v \in (\mathcal{V}_h)^d$ then $\Pi v = v$ and $a(\cdot, \cdot)$ is reduced to a more familiar fashion.

Multiscale method

Motivation

In many applications, solution exist on several different scales e.g. flow in porous media and in composite materials.

- Secondary oil recovery.
- Sequestration of Carbon Dioxide.

Why do we need to resolve the coefficients?

Example with periodic coefficient

Consider Possion's equation with period coefficient $\alpha = \alpha(x/\epsilon)$. For the finite element method, we have

$$||\sqrt{\alpha}\nabla(u - u_h)||_{L^2(\Omega)} \leq C\frac{H}{\epsilon}||f||_{L^2(\Omega)}$$

- Need $H \ll \epsilon$ for reliable results.
- To computational expensive to solve on a single mesh for many applications e.g. flow in porous media and in composite materials.
- Want eliminate the ϵ dependence by using a multiscale method (Målqvist-Peterseim).

Framework for Multiscale methods

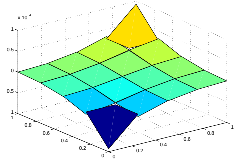
The problem is split into one coarse and fine scale contribution $\mathcal{V}_h = \mathcal{V}_c \oplus \mathcal{V}_f$.

- Let subdivide Ω into a coarse mesh $\mathcal{K}_c = \{K_c\}$.
- $\mathcal{V}_c = span\{\phi_i\} = \Pi_c \mathcal{V}_h$ and $\mathcal{V}_f = \{v \in \mathcal{V}_h : \ \Pi_c v = 0\}$, where $\Pi_c : \mathcal{V}_h \to \mathcal{V}_c$ is the L^2 projection onto the coarse mesh.
- Define the map $\mathcal{T} : \mathcal{V}_c \to \mathcal{V}_f$ as

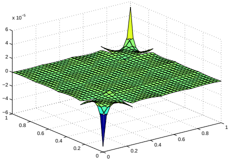
$$a(\mathcal{T} v_c, v_f) = -a(v_c, v_f), \quad \forall v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$

Split $u_h = u_c + \mathcal{T} u_c + u_f$ and $v = v_c + v_f$ where $u_c \in \mathcal{V}_c$, $v_f \in \mathcal{V}_f$.

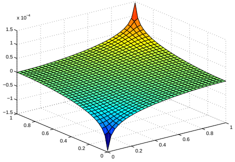
$$a(u_c + \mathcal{T} u_c + u_f, v_c + v_f) = l(v_c + v_f), \quad \forall v_c \in \mathcal{V}_c, v_f \in \mathcal{V}_f$$



(c) u_c



(d) $\mathcal{T} u_c + u_f$



(e) $u_h = u_c + \mathcal{T} u_c + u_f$

Fine scale

Let $v_c = 0$ to get the fine scale equations

$$a(\mathcal{T} u_c + u_f, v_f) = l(v_f) - a(u_c, v_f),$$

split into two equations

$$\begin{aligned} a(u_f, v_f) &= l(v_f) \quad \forall v_f \in \mathcal{V}_f, \\ a(\mathcal{T} u_c, v_f) &= -a(u_c, v_f) \quad \forall v_f \in \mathcal{V}_f. \end{aligned}$$

Coarse scale

Let $v_f = 0$ on the coarse scale

$$a(u_c + \mathcal{T} u_c, v_c) = l(v_c) - a(u_f, v_c) \quad \forall v_c \in \mathcal{V}_c$$

Comments

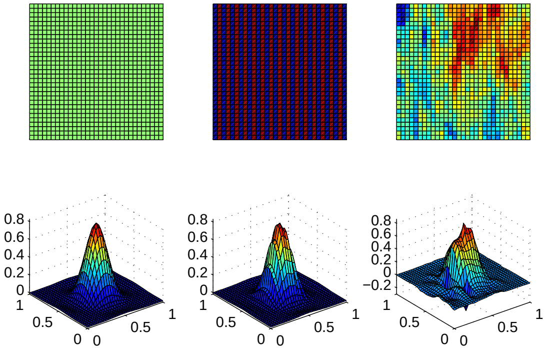
- Equally hard to solve as the original problem.
- Other chooses then Π_c can be coincided.
- A symmetric split can also be considered for the coarse scale problem.

View solution as span av modified basis functions

- Let $\mathcal{V}_c = \text{span}\{\phi_i\}$ and $\mathcal{V}^{ms} = \text{span}\{\phi_i + \mathcal{T}\phi_i\}$.
- View $\phi_i + \mathcal{T}\phi_i$ as a modified basis function.

From the multiscale map we have, $\mathcal{V}_h = \mathcal{V}_{ms} \perp_a \mathcal{V}_f$, for all i

$$a(\phi_i + \mathcal{T}\phi_i, v) = 0, \qquad v \in \mathcal{V}_f$$



Approximation of $\mathcal{T}\phi_i$

- The fast decay of $\mathcal{T}\phi_i$ motivates approximations of $\mathcal{T}\phi_i$ to patches $\omega_i^L \subset \Omega$.

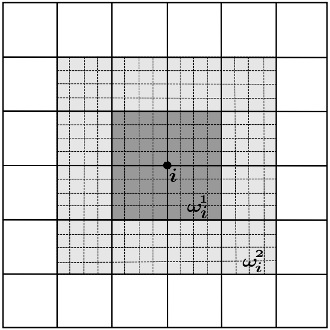
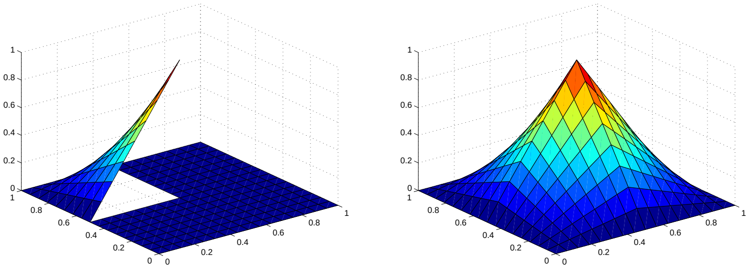


Figure: Example of a one layer patch ω_i^1 and a two layer patch ω_i^2

Multiscale method discretization

- $\tilde{\mathcal{T}}$ is he restriction of $\mathcal{T}$ to a patch $\omega \subset \Omega$
- $\tilde{U}_f = \sum_{i \in \mathcal{N}} \tilde{U}_{f,i}$ where $\mathcal{N}$ is the number of nodes, be the approximation of u_f.
- Let $\mathcal{M}_i$ be all j s.t $\phi_j = 1$ in node i.
- Let also $\Phi_i = \sum_{j \in \mathcal{M}_i} \phi_j$



(a) ϕ_j

(b) $\Phi_i = \sum_{j \in \mathcal{M}_i} \phi_j$

Bilinear form for the fine scale problem

- Let $\mathcal{K}(\omega_i^L) = \{K : K \cap \omega_i^L \neq 0\}$.
- Let also $\Gamma^I(\omega_i^L)$ be all interior edges on $\mathcal{K}(\omega_i^L)$.

Define $a_i : \mathcal{V}_f(\omega_i) \times \mathcal{V}_f(\omega_i) \to \mathbb{R}$, as

$$
\begin{aligned}
a_i(v, z) = \sum_{K \in \mathcal{K}(\omega_i^L)} (\alpha \nabla v, \nabla z)_{L^2(K)} - \sum_{e \in \Gamma^I(\omega_i^L)} \Big((\mathbf{n} \cdot \{\alpha \Pi \nabla v\}_w, [z])_{L^2(e)} \\
+ (\mathbf{n} \cdot \{\alpha \Pi \nabla z\}_w, [v])_{L^2(K)} - \frac{\sigma_e \gamma_e}{h_e} ([v], [z])_{L^2(e)} \Big), \\
l_i(v) = (\Phi_i f, v)_{L^2(\Omega)}.
\end{aligned}
$$

Fine scale equations

For all $i \in \mathcal{N}$: find $\tilde{\mathcal{T}}\phi_j \in \mathcal{V}_f(\omega_i^L)$ and $U_{f,i} \in \mathcal{V}_f(\omega_i^L)$ for $j \in \mathcal{M}_i$ s.t

$$a_i(\tilde{\mathcal{T}}\phi_j, v_f) = -a_i(\phi_j, v_f), \quad \forall v_f \in \mathcal{V}_f(\omega_i^L),$$
$$a_i(\tilde{U}_{f,i}, v_f) = l_i(\Phi_i v_f), \quad \forall v_f \in \mathcal{V}_f(\omega_i^L).$$

Coarse scale equation

Find $U_c \in \mathcal{V}_c$ s.t

$$a(U_c + \tilde{\mathcal{T}} U_c, v_c) = l(v_c) - (\tilde{U}_f, v_c), \quad \forall v_c \in \mathcal{V}_c.$$

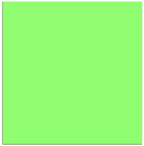
Decay in V_f

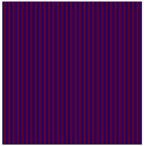
Problem setting

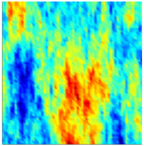
- Let the computational domain be ω_i^L for $L = 1, 2, \ldots, N$ where $\omega_i^L \subseteq \Omega$.
- Let also $\Phi_i = \sum_{j \in \mathcal{M}_i} \phi_j$
- The problem reads: find $\tilde{\mathcal{T}}\Phi_i \in \mathcal{V}_h(\omega_i^L)$

$$a(\tilde{\mathcal{T}}\Phi_i, v) = -a(\Phi_i, v), \quad \forall v \in \mathcal{V}_h(\omega_i^L).$$

- The reference solution $\mathcal{T}\Phi_i$ is the solution computed on $\omega_i^N = \Omega$.
- Coarse mesh is 8×8 element and reference grid is 64×64 elements.







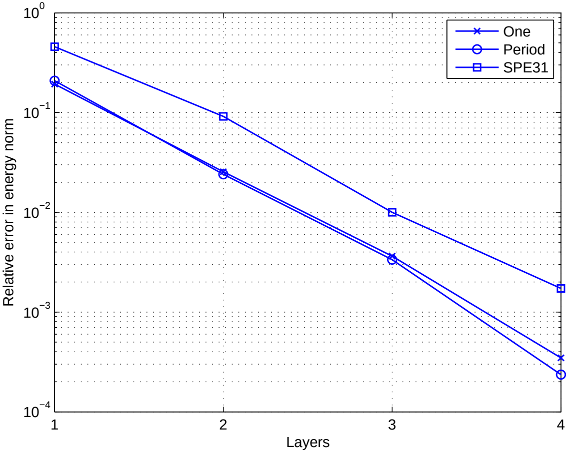


Figure: The error in relative error in broken energy norm with respect to the path size.

Convergence

Problem setting

- Consider the model problem (Poisson's equation)
- Keeping the refinement level constant and increasing the patch sizes $L = 1, \ldots, N$ for all local problems.
- The coarse grid is 8×8 coarse elements.
- The reference solution U_{ref} is the DG solution computed on 64×64 elements.
- The right hand side is -1 in the lower left corner and 1 in the upper right.

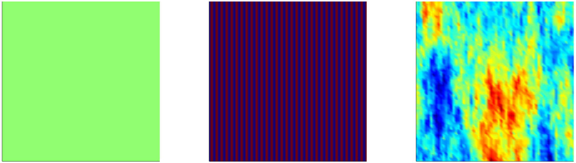


Figure: Permeabilities α.

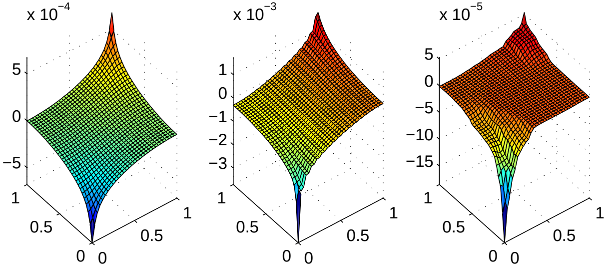


Figure: The reference solution to the model problem using the permeabilities *One*, *Period* and *SPE*

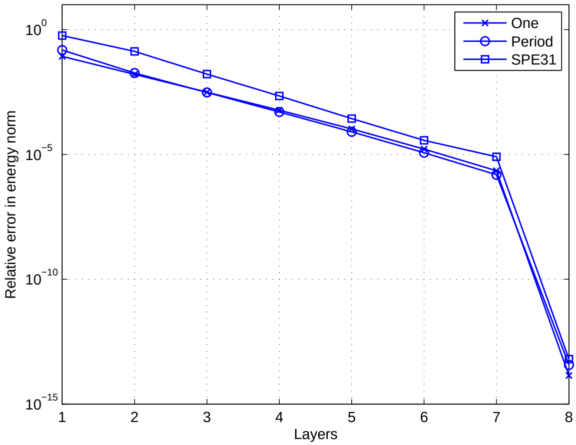


Figure: The relative error in broken energy norm with respect to the patch sizes.

Implementation

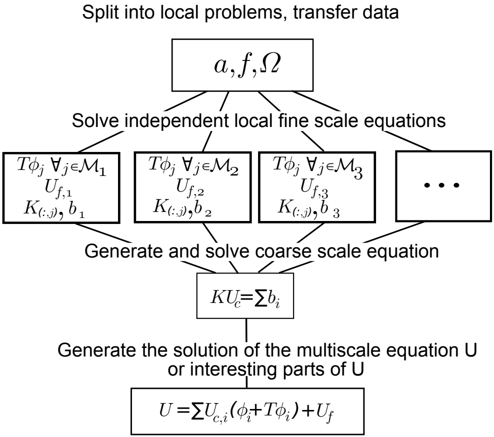


Figure: Scheme of the implementation.

Constraints on the fine scale equations

- The condition is realised using Lagrangian multiplier.
- Let ϕ be a coarse basis function and φ be a fine basis function.

Want so fined $\tilde{\mathcal{T}} w \in \mathcal{V}_f(\omega_i^L)$

$$a_i(\tilde{\mathcal{T}} w, v) = -a_i(w, v) \quad \forall v \in \mathcal{V}_f(\omega_i^L).$$

Algebraic problem reads:

$$\begin{pmatrix} K & P^T \\ P & 0 \end{pmatrix} \xi = \begin{pmatrix} b \\ 0 \end{pmatrix},$$

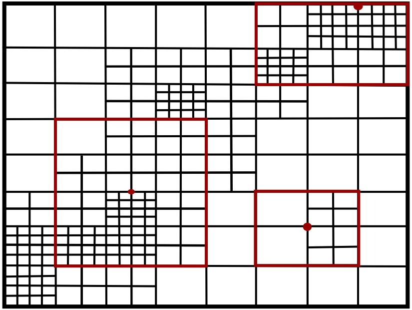
where $K_{k,l} = a_i(\varphi_k, \varphi_l)$, $b_k = -a_i(\phi_l, \varphi_k)$ and

$$P = \begin{pmatrix} (\phi_1, \varphi_1) & (\phi_1, \varphi_2) & \dots & (\phi_1, \varphi_N) \\ (\phi_2, \varphi_1) & (\phi_2, \varphi_2) & \dots & (\phi_2, \varphi_N) \\ \vdots & \vdots & \ddots & \vdots \\ (\phi_M, \varphi_1) & (\phi_M, \varphi_2) & \dots & (\phi_M, \varphi_N) \end{pmatrix}.$$

Adaptivity

Set up

- Need an a posteriori error estimate for the discontinuous Galerkin method.
- Use this in the framework for Multiscale methods to construct a a posteriori error estimate for the multiscale method.
- Construct a adaptive algorithm to automatically tune the critical parameters.



Theorem (A posteriori error estimate for DG method)

- *Let u, u_h be given by the exact solution respectively the DG solution.*
- *Let also $\chi \in \mathcal{V}_h \cap H^1(\Omega)$*
- *Moreover, let $\mathcal{E} := \mathcal{E}_c + \mathcal{E}_d$ where $\mathcal{E}_c := u - \chi$ and $\mathcal{E}_d := \chi - u_h$.*

Then,

$$\sum_{K \in \mathcal{K}} ||\sqrt{\alpha} \nabla \mathcal{E}_c||^2_{L^2(K} \lesssim \sum_{K \in \mathcal{K}} (\varrho_K(u_h) + \zeta_K(u_h, \chi))^2,$$

where

$$\begin{aligned} \varrho_K(u_h) &= \frac{h_K}{\sqrt{\alpha_0}} ||f + \nabla \cdot \alpha \nabla u_h||_{L^2(K)}, \\ &+ \sqrt{\frac{h_K}{\alpha_0}} \Big(||(1 - w_{K(e)}) n \cdot [\alpha \nabla u_h]||_{L^2(\partial K)} + ||\frac{\sigma_e \gamma_e}{h_e} [u_h]||_{L^2(\partial K \setminus \Gamma^B)} \Big), \\ \zeta_K(u_h, \chi) &= ||\sqrt{\alpha} \nabla (u_h - \chi)||_{L^2(K)}. \end{aligned}$$

Treatment of $\zeta_K(u_h, \chi)$

1. First, χ has to be chosen in a clever way.
2. Second, $\zeta_K(u_h, \chi)$ can either be estimated or evaluated.

- One possible chose is $\chi = \mathcal{I}_{Os} u_h$.
- Under certain assumption on α, $\zeta_K(u_h, \chi)$ can be evaluated and hidden in $\varrho_K(u_h)$.

Lemma (Oswald interpolation operator)

- *Let* $\mathcal{I}_{Os} : \mathcal{V}_h \to \mathcal{V}_h \cap H^1$

$$\mathcal{I}_{Os} v = \sum_{i \in \mathcal{N}} \left(\frac{1}{|\mathcal{M}_j|} \sum_{j \in \mathcal{M}_i} v_j(x_i) \varphi_j \right).$$

Then,

$$||\sqrt{\alpha} \nabla (v - \mathcal{I}_{Os} v)||^2 \lesssim \alpha^0 || \frac{1}{\sqrt{h_e}} [v] ||^2_{L^2(\partial K \setminus \Gamma^B)}.$$

Sketch of proof

We have

$$\sum_{K\in\mathcal{K}} \|\sqrt{\alpha}\nabla\mathcal{E}_c\|^2_{L^2(K)} = a(\mathcal{E}_c,\mathcal{E}_c) = a(\mathcal{E},\mathcal{E}_c) - a(\mathcal{E}_d,\mathcal{E}_c),$$

where

$$\begin{aligned} a(\mathcal{E},\mathcal{E}_c) &= a(u,\mathcal{E}_c) - a(u_h,\mathcal{E}_c) = l(\mathcal{E}_c) - a(u_h,\mathcal{E}_c),\\ &= l(\eta) - a(u_h,\eta), \end{aligned}$$

where $\eta = \mathcal{E}_c - \pi_0\mathcal{E}_c$.

- First integration by parts $l(\eta) - a(u_h,\eta)$ element wise and using the identity $[vz] = \{v\}_w[z] + \{v\}_{\bar{w}}[z]$.

We get,

$$l(\eta) - a(u_h, \eta)$$
$$= \sum_{K \in \mathcal{K}} (f + \nabla \cdot \alpha \nabla u_h, \eta)_{L^2(K)} + \sum_{e \in \Gamma^I} \Big(-(n \cdot [\alpha \nabla u_h], \{\eta\}_{\bar{w}})_{L^2(e)}$$
$$+ (n \cdot \{\alpha \Pi \nabla \eta\}_w, [u_h])_{L^2(e)} - \sigma \gamma_e h_e^{-1} ([u_h], [\eta])_{L^2(e)} \Big) + \sum_{e \in \Gamma^B} (n \cdot \alpha \nabla u_h, \eta)_{L^2(e)}.$$

1. Then, using the inequalities and stability for the piecewise constant L^2-projection.

$$||v - \pi_0 v||_{L^2(K)} \lesssim \frac{h_K}{\sqrt{\alpha_0}} ||\sqrt{\alpha} \nabla v||_{L^2(K)}, \quad \forall v \in H^1(K),$$
$$||v - \pi_0 v||_{L^2(\partial K)} \lesssim \sqrt{\frac{h_K}{\alpha_0}} ||\sqrt{\alpha} \nabla v||_{L^2(K)} \quad \forall v \in H^1(K).$$

2. For $a(\mathcal{E}_d, \mathcal{E}_c)$ use the Lemma (Oswald interpolation operator).

Theorem (A posteriori error estimate for ADG-MS)

- *Let u, U be the exact solution respectively the multiscale solution.*
- *Let* $X = \mathcal{I}_{Os} U \in H^1(\Omega)$.
- *Set* $\mathscr{E} := \mathscr{E}_c + \mathscr{E}_d$ *where* $\mathscr{E}_c := u - X$ *and* $\mathscr{E}_d := X - U$.
- $U_i := \sum_{j \in \mathcal{M}_i} U_{c,j}(\phi_j + \tilde{\mathcal{T}}\phi_j) + U_{f,i}$, *where* $U_{c,j}$ *are the nodal values.*

Then,

$$\sum_{K \in \mathcal{K}} ||\sqrt{\alpha}\nabla \mathscr{E}_c||^2_{L^2(K)} \lesssim \sum_{K_c \in \mathcal{K}_c} \rho^2_{h,K_c} + \sum_{i \in \mathcal{N}} \rho^2_{L,\omega_i^L},$$

where

$$\rho^2_{L,\omega_i^L} = \sum_{e \in \Gamma^B(\omega_i^L) \setminus \Gamma^B} \rho^2_{L,\omega_i^L,e},$$

$$\rho_{L,\omega_i^L,e} = \frac{H_{\omega_i^L}}{\sqrt{h_K \alpha_0}} \Big(||n \cdot \{\alpha \nabla U_i\}_w||_{L^2(e)} + \frac{\sigma_e \gamma_e}{h_e} ||[U_i]||_{L^2(e)} \Big),$$

measures the effect of the truncated patches.

Also

$$\rho_{h,K_c}^2 = \sum_{K \in K_c} (\varrho_K(U) + \zeta_K(U,X))^2,$$

with ϱ_K and ζ_K as in previous theorem.

Comments

- $\rho_{L,\omega_i^L}^2$ measure the effect of the truncated patches.
- $\rho_{h,K}^2$ measure the effect of the refinement level.
- $\frac{1}{\sqrt{h_K}h_e}|||[U_i]|||_{L^2(e)}$ behave as $h^{-3/2}e^{-L} \sim 1 \Rightarrow L \sim \frac{3}{2}\log h^{-1}$
- Another possible choice is a weighted Oswald-type interpolation operator with the weights depending on the diffusion tensor.

Sketch of proof

We have

$$a(\mathscr{E}_c, \mathscr{E}_c) = a(\mathscr{E}, \mathscr{E}_c) - a(\mathscr{E}_d, \mathscr{E}_c),$$

where

$$\begin{aligned} a(\mathscr{E}, \mathscr{E}_c) &= a(u, \mathscr{E}_c) - a(U, \mathscr{E}_c), \\ &= l(\mathscr{E}_c) - a(U, \mathscr{E}_c), \\ &= l(\mathscr{E}_c - v_c) - a(U, \mathscr{E}_c - v_c), \\ &= \sum_{i \in \mathcal{N}} \Big(l_i(\mathscr{E}_c - v_c - v_f) - a(U_i, \mathscr{E}_c - v_c) + a_i(U_i, v_f) \Big). \end{aligned}$$

here $v_c \in \mathcal{V}_c$ and $v_f \in \mathcal{V}_f$.
Notice that

$$\begin{aligned} a_i(\mathscr{E}, \mathscr{E}_c) = a(\mathscr{E}, \mathscr{E}_c) + \sum_{e \in \Gamma^B(\omega_i^L)\Gamma^B} \Big(&(n \cdot \{\alpha \nabla U_i\}_w, [v_f])_{L^2(e)} \\ &+ (n \cdot \{\alpha \nabla U_i\}_w, [v_f])_{L^2(e)} - \frac{\sigma_e \gamma_e}{h_e}([U_i], [v_f])_{L^2(e)} \Big) \end{aligned}$$

Then,

$$
\begin{aligned}
a(\mathscr{E},\mathscr{E}_c) &= \Big(l(\mathscr{E}_c - v_c - v_f) - a(\mathscr{E},\mathscr{E}_c - v_c - v_f) \Big) \\
&+ \sum_{i\in\mathcal{N}} \sum_{e\in\Gamma^B(\omega_i^L)\Gamma^B} \Big((n\cdot\{\alpha\nabla U_i\}_w, [v_f])_{L^2(e)} + (n\cdot\{\alpha\nabla U_i\}_w, [v_f])_{L^2(e)} \\
&- \frac{\sigma_e\gamma_e}{h_e}([U_i],[v_f])_{L^2(e)} \Big) \\
&=: I + II.
\end{aligned}
$$

1. The first term (I), is bounded by the a apoteriori error estimate for DG.
2. To bound the second term (II),
   - Select v_c and v_f as the piecewise constant L^2-projection onto V_c and V_f, respectively.
   - Then using a trace inequality, a interpolation estimate and L^2-stability of π_f, II is bounded.

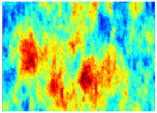
Adaptive algorithm

Algorithm 1 Adaptive Discontinuous Galerkin Multiscale Method

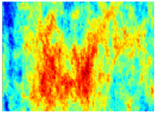
Initialize the coarse mesh with mesh size H.
Let the fine mesh size be $h_K = H/2$ for all $K_c \in \mathcal{K}_c$ and $L(\omega_i) = 2$ for all $i \in \mathcal{N}$
while $\sum_{i \in \mathcal{N}} (\rho^2_{h,\omega_i} + \rho^2_{L,\omega_i}) > TOL$ **do**
    for $i \in \mathcal{N}$ **do**
        if $\rho^2_{L,\omega_i} > TOL/(2\mathcal{N})$ **then**
            $L(\omega_i) := L(\omega_i) + 1$
        end if
    end for
    for $K_c \in \mathcal{K}_c$ **do**
        if $\rho^2_{h,K} > TOL/(2|\mathcal{K}_c|)$ **then**
            $h_K := h_K/2$
        end if
    end for
end while

Adaptivity

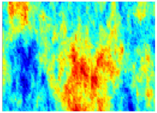
- Consider the model problem
- Using the a posteriori error estimate to construct an adaptive algorithm.
- Start with one refinement and 2 layers patches everywhere.
- Refine 30% of the coarse elements and increase 30% of the patch sizes in each iteration.
- Coarse mesh is 32×32 element and reference grid is 256×256 elements.
- The right hand side is -1 in the lower left corner and 1 in the upper right.



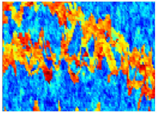
(a) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$



(b) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$



(c) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^5$



(d) $\frac{\alpha_{max}}{\alpha_{min}} \sim 10^6$

Figure: Permeabilities α projection in log scale.

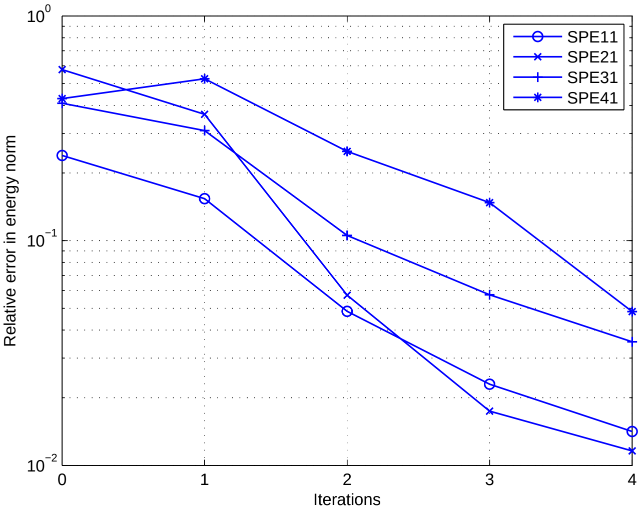
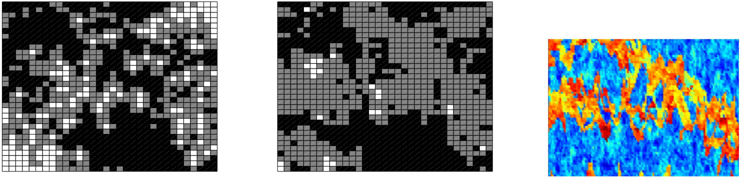


Figure: The relative error in broken energy norm with respect to number of iterations. Iteration 0 corresponds to the standard DG solution and iteration 1 the start values in the adaptive algorithm.

Figure (a) and (b) illustrates where the adaptive algorithm puts most effort

- Figure (a) corresponds to the refinements
- Figure (b) corresponds to the patch sizes.
- Figure (c) is the permeability α.



(a) Refine h_K (b) Layers, L (c) Layers, L

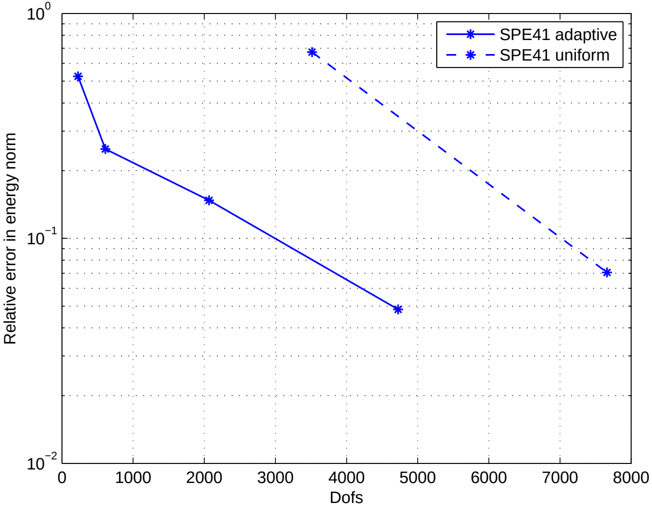


Figure: The relative error in broken energy norm with respect to the mean value of the degrees of freedom for the fine scale problems.

Conclusions

Advantage

- The fine scale problems are perfectly parallelizable.
- The exponential decay in the fine scale solution allows small patches.
- The error estimate and the adaptivity algorithm focus computational effort in critical areas.
- Very high aspect ratio in α can be solved.
- Possible to construct a conservative flux on the coarse scale.

Future work

- A priori analysis of the discontinuous Galerkin multiscale method.
- Hybrid method using DG on coarse scale and CG on fine scale.
- Extend the method and analysis to diffusion-convection problems.
- Investigate to sensibility in input data (uncertainty in the permeability α)
- Extend the implementation to triangular meshed to allow for complicated geometries.
- 3D implementation.

Questions