Performance Analysis, Autumn 2010

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Queueing Networks

Modeling a queue of jobs

- Helpful fact: Burke's theorem:

  The departure process from a stable $M/M/c$ queue with arrival and service rates λ and μ, respectively, is a Poisson process with rate λ.

- This allows to analyze acyclic QNs of queues with exponential service times, since each queue will behave like an $M/M/1$ ($M/M/c$) queue

Feed-Forward Queueing Networks

QN of queues with exponential service times, where no cycles.

- Let r_{ij} be routing probabilities, i.e., probability that job leaving i will go to j.
- Assume external input comes with rate λ_0.
- r_{0j} is fraction of arrivals that come to j,
- r_{i0} is fraction of departures from i that leave network.
- Solve the *traffic equations*:

$$\lambda_i = \lambda_0 r_{0i} + \sum_{j=1}^{k} \lambda_j r_{ji}$$

- Each queue i will behave like $M/M/1$, with arrival rate $\lambda_0 r_{0i}$ and service rate μ_i,
- Assume no bottlenecks, i.e., $\dfrac{\lambda_i}{\mu_i} = \rho_i < 1$,

Jackson Queueing Networks

Jackson QN: a QN of queues with exponential service times, where cycles are allowed: It should be *open*, i.e., accept jobs from environment.

- Now arrival streams may not be Poisson.
- But steady-state probabilities are as if each queue is independent $M/M/1$ queue
- Assume external input comes to queue 0 with rate λ_0.
- Solve the *traffic equations*:

$$\lambda_i = \lambda_0 r_{0i} + \sum_{j=1}^{k} \lambda_j r_{ji}$$

Jackson Queueing Networks (ctd.)

Product form solution: In a Jackson QN, steady state probability distributions are product of individual queue distributions:

- Let $\overline{n} = n_1 n_2 n_3 \ldots n_k$,
- let $\pi_{\overline{n}} = P(\bigwedge_{i=1}^{k} \text{queue } i \text{ has } n_i \text{ elements})$.
- Then

$$\pi_{\overline{n}} = \prod_{i=1}^{k} P(\text{queue } i \text{ has } n_i \text{ elements}) = \prod_{i=1}^{k} (1 - \rho_i)\rho_i^{n_i}$$

- Sometimes

$$\pi_{\overline{n}} = \frac{1}{G} \prod_{i=1}^{k} \rho_i^{n_i}$$

Closed Queueing Networks

Gurdon Newell QNs: queues with exponential service times, which is *closed*, i.e., no jobs from environment.

- Let K be total number of customers (fixed population)
- Let queues be numbered $1, 2, \ldots, k$,
- Now, traffic equations $\lambda_i = \sum_{j=1}^{k} \lambda_j r_{ji}$ have rank $k-1$.

  Typically normalize wrp. to λ_1. Let $\gamma_i = \lambda_i / \lambda_1$.
- Also, relativize ρ_i by $\rho_i = \gamma_i / \mu_i$.
- Then again

$$\pi_{\overline{n}} = \frac{1}{G(k,K)} \prod_{i=1}^{k} \rho_i^{n_i},$$

  but now we cannot break apart $G(k, K)$ due to interdependence on number of jobs in each queue (fixed total population).

Closed Queueing Networks: example

Consider a GNQN with 3 stations (1, 2, and 3).

- $\mu_1 = 1$, $\mu_2 = 1/2$, and $\mu_3 = 1/3$,
- $r_{12} = 0.4$, $r_{13} = 0.6$, $r_{21} = r_{31} = 1$
- Let $K = 3$ (3 jobs circulating).
- Solve traffic equations, yields: $\gamma_1 = 1$, $\gamma_2 = 0.4$, $\gamma_3 = 0.6$
- Get service demands $\rho_1 = 1$, $\rho_2 = 0.8$, $\rho_3 = 1.8$
- There are 10 states.

Closed Queueing Networks: example (ctd.)

- $G(3,3) = \sum_{n_1+n_2+n_3=3} \rho_1^{n_1} \rho_2^{n_2} \rho_3^{n_3} = 19.008$
- For instance $\pi_{111} = \rho_1 \rho_2 \rho_3 / G(3,3) = 1.44/19.008 = 0.0758$
- $P(n_1 = 0) = \frac{1}{19.008}(\rho_2^3 + \rho_3^3 + \rho_2 \rho_3^2 + \rho_2^2 \rho_3) = \frac{10.0880}{19.008} = 0.5307$
- Thus, througput at node 1 is $\lambda_1 = P(n_1 \neq 0)\mu_1 = 0.4693 \times 1 = 0.4693$.
- Average number of jobs at queue 1:

$$E[L_1] = \frac{\rho_1(\rho_2^2 + \rho_3^2 + \rho_2 \rho_3) + 2\rho_1^2(\rho_2 + \rho_3) + 3\rho_1^3}{19.008} = 0.7113$$

- Average response time at queue 1 (by Little's law):

$$E[W_1] = \frac{E[L_1]}{\lambda_1} = \frac{0.7113}{0.4693} = 1.516$$

Mean-value Analysis

A faster computation for getting average numbers for a GNQN.

- Helpful fact: Arrival theorem:

  In a closed QN, steady-state prob. distr. of jobs at the instant that a job moves from one queue to another, equals the (usual) steady-state prob. distr. of jobs in that QN without the moving job.

- Let average waiting time, etc. depend on K: e.g., $E[W_i](K)$.
- By arrival theorem:

$$E[W_i](K) = \frac{1 + E[L_i](K-1)}{\mu_i}$$

Mean-value Analysis (ctd.)

- By Little's law:

$$E[L_i](K) = \lambda_i(K)E[W_i](K) = \lambda_1(K)\gamma_i E[W_i](K)$$

- Now:

$$\sum_{i=1}^{k} E[L_i](K) = E\left[\sum_{i=1}^{k} L_i\right](K) = K$$

- i.e.,:

$$\lambda_1(K) = \frac{K}{\sum_{j=1} \gamma_j E[W_j](K)}$$

- i.e.,:

$$E[L_i](K) = K \frac{\gamma_i E[W_i](K)}{\sum_{j=1} \gamma_j E[W_j](K)}$$

Mean-value Analysis: summary

MVA has the following steps:

- Solve traffic equations, to obtain γ_i (let $\gamma_1 = 1$)
- For $K = 1, 2, \ldots,$ calculate:

$$
\begin{aligned}
E[W_i](K) &= \frac{1 + E[L_i](K-1)}{\mu_i} \\
E[L_i](K) &= K \frac{\gamma_i E[W_i](K)}{\sum_{j=1} \gamma_j E[W_j](K)}
\end{aligned}
$$

- Start by: $E[L_i](0) = 0$, i.e., $E[W_i](1) = \frac{1}{\mu_i}$.

Mean-value Analysis (example)

Back to example:

K	$\lambda_1(K)$	$E[W_1](K)$	$E[W_2](K)$	$E[W_3](K)$
1	0.278	1	2	3
2	0.404	1.278	2.444	4.500
3	0.469	1.516	2.790	6.273

K	$\lambda_1(K)$	$E[L_1](K)$	$E[L_2](K)$	$E[L_3](K)$
1	0.278	0.278	0.222	0.500
2	0.404	0.516	0.395	1.091
3	0.469	0.711	0.523	1.765