Performance Analysis, Autumn 2010

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Exponential Distribution

Properties of the Exponential Distribution, with parameter λ

- Probability density function: $\lambda e^{-\lambda t}$
- cumulative distribution function $P(X \leq t) = 1 - e^{-\lambda t}$
- mean

$$\int_0^\infty t\lambda e^{-\lambda t} dt = [-te^{-\lambda t}]_{t=0}^{t=\infty} + \int_0^\infty e^{-\lambda t} dt$$

$$[-\tfrac{1}{\lambda} e^{-\lambda t}]_{t=0}^{t=\infty} = \tfrac{1}{\lambda}$$

- Memoryless property:

$$P[X \leq t' | X \geq t] = \tfrac{P(t \leq X \leq t')}{P(X \geq t)} =$$

$$\tfrac{e^{-\lambda t} - e^{-\lambda t'}}{e^{-\lambda t}} = 1 - e^{-\lambda (t' - t)} = P(X \leq (t' - t))$$

Exponential Distribution

More Properties of the Exponential Distribution:

- minimum of two independent exponentially distributed random variables X_1 and X_2 with parameters λ_1 and λ_2:

$$P(X_1 \leq t \vee X_2 \leq t) = 1 - P(X_1 > t \wedge X_2 > t) =$$

$$1 - (P(X_1 > t)P(X_2 > t)) =$$

$$1 - e^{-\lambda_1 t} e^{-\lambda_2 t} = 1 - e^{-(\lambda_1 + \lambda_2)t}$$

Exponential Distribution

More Properties of the Exponential Distribution:

- Probability that X_1 will "win" over X_2:

$$P(X_1 \leq X_2) = \int_0^\infty Pr(X_2 > t)\lambda_1 e^{-\lambda_1 t} dt =$$

$$\lambda_1 \int_0^\infty e^{-\lambda_2 t} e^{-\lambda_1 t} dt =$$

$$\lambda_1 \int_0^\infty e^{-(\lambda_1+\lambda_2)t} dt = \frac{\lambda_1}{\lambda_1 + \lambda_2}$$

Continuous-Time Markov Chains

A **Continuous-Time Markov chain** is a continuous-time and discrete-state Markov process.

- Time domain $\mathcal{R}^{\geq 0}$,
- A set S of *states*, usually $\{0, 1, 2, 3. \dots\}$ or $\{0, 1, 2, \dots, n\}$,
- Probability mass function at time t: $\pi_i(t) = P(X_t = i)$, as Vector: $\pi(t) = (\pi_0(t)\pi_1(t) \dots)$
- Initially: $\pi_i(0) = P(X_0 = i)$ with $\sum_{i \in S} \pi_i(0) = 1$

Continuous-Time Markov Chains

Assume MC is *time-homogeneous*.

- Let $p_{ij}\Delta t$ be the probability to move from state i to state j in a short interval Δt, i.e.,

$$P[X_{(t+\Delta t)} = j | X_t = i] \approx p_{ij}\Delta t$$

- Let $p_i\Delta t$ be probability to move away from state i in a short interval Δt, i.e.,

$$P[X_{(t+\Delta t)} = i | X_t = i] \approx 1 - p_i\Delta t$$

- Define $p_{ii} = -p_i$
- Then $\pi_i(t+\Delta t) \approx \pi_i(t) + \sum_{j \in S} \pi_j(t) p_{ji} \Delta t$
- i.e., $\frac{d\pi_i}{dt} = \sum_{j \in S} \pi_j(t) p_{ji}$

Continuous Time Markov Chain

Definition of Continuous Time Markov chain

- An *initial probability distribution* $\pi^{(0)}$ with
- an *intensity matrix*

$$P = \begin{pmatrix} -p_0 & p_{01} & p_{02} & \cdots \\ p_{10} & -p_1 & p_{12} & \cdots \\ p_{20} & p_{21} & -p_2 & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with $\sum_{j \neq i} p_{ij} = p_i$ for each $i \in S$.

When the Markov chains reaches a state i, it stays there for some exponentially distributed time with mean $\frac{1}{p_i}$. When it leaves state i, it jumps to state j (with $j \neq i$) with probability $\frac{p_{ij}}{p_i}$, and the behavior continues as before.

Simple CTMC example

Modeling a queue of jobs

- Initially the queue is empty
- Jobs arrive with rate 3/2 (i.e., mean inter-arrival time is 2/3)
- Jobs are served with rate 3 (i.e., mean service time is 1/3)
- maximum size of the queue is 3.

Intensity matrix:

$$Q = \begin{bmatrix} -3/2 & 3/2 & 0 & 0 \\ 3 & -9/2 & 3/2 & 0 \\ 0 & 3 & -9/2 & 3/2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

Embedded Discrete Time Markov Chain

Focus only on the state *changes*
Example:

$$\text{from} \quad Q = \begin{bmatrix} -3/2 & 3/2 & 0 & 0 \\ -3 & -9/2 & 3/2 & 0 \\ 0 & 3 & -9/2 & 3/2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

derive

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 2/3 & 0 & 1/3 & 0 \\ 0 & 2/3 & 0 & 1/3 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

CTMC: Transient probabilities

How to derive $\pi(t)$, probability distribution at time t?

- Remember: $\frac{d\pi(t)}{dt} = \sum_{j \in S} \pi(t) p_{ji} = \pi Q$
- So: $\pi(t)$ is obtained by solving matrix differential equation.
- The stationary distribution should satisfy

$$\frac{d\pi(t)}{dt} = \pi Q = 0$$

Steady-state for Example

- Intensity matrix:

$$Q = \begin{bmatrix} -3/2 & 3/2 & 0 & 0 \\ 3 & -9/2 & 3/2 & 0 \\ 0 & 3 & -9/2 & 3/2 \\ 0 & 0 & 3 & -3 \end{bmatrix}$$

- We solve the equations: $\pi Q = 0$
- Solution: $\pi = (8/15, 4/15, 2/15, /15)$

Classification of states:

States in an CTMC can be of three types.

- Define $T_i = \min\{t > 0 \ : \ X_t = i\}$ (time to reach i).
- state i is *transient* if $P[T_i < \infty | X_0 = i] < 1$,
- state i is *recurrent* if $P[T_i < \infty | X_0 = i] = 1$,
  - state i is *positive recurrent* if $E[T_i | X_0 = i] < \infty$,
  - state i is *null recurrent* if $E[T_i | X_0 = i] = \infty$,
- Fact: in an irreducible MC, all states are of the same type (either transient, null recurrent, or positive recurrent).

Birth-Death Processes

Consider a CTMC $\langle S, \pi(0), Q \rangle$, with:

- $S = \{0, 1, 2, 3, \ldots\}$
-

$$Q = \begin{bmatrix} -\lambda_0 & \lambda_0 & 0 & 0 & 0 & \cdots \\ \mu_1 & -(\mu_1 + \lambda_1) & \lambda_1 & 0 & 0 & \cdots \\ 0 & \mu_2 & -(\mu_2 + \lambda_2) & \lambda_2 & 0 & \cdots \\ 0 & 0 & \mu_3 & -(\mu_3 + \lambda_3) & \lambda_3 & \cdots \\ 0 & 0 & 0 & \mu_4 & -(\mu_4 + \lambda_4) & \cdots \\ 0 & 0 & 0 & 0 & \mu_5 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

- Finding steady-state distribution π by equations

$$\begin{cases} (\lambda_n + \mu_n)\pi_n & = & \lambda_{n-1}\pi_{n-1} + \mu_{n+1}\pi_{n+1} \qquad (n \geq 1) \\ \lambda_0 \pi_0 & = & \mu_1 \pi_1 \end{cases}$$

Birth-Death Processes (cont.)

- Solving

$$\begin{cases} (\lambda_n + \mu_n)\pi_n = \lambda_{n-1}\pi_{n-1} + \mu_{n+1}\pi_{n+1} & (n \geq 1) \\ \lambda_0\pi_0 = \mu_1\pi_1 \end{cases}$$

- Assume π_0 given
- $\pi_1 = \frac{\lambda_0}{\mu_1}\pi_0$
- $\pi_2 = \frac{\lambda_1\lambda_0}{\mu_2\mu_1}\pi_0$
- In general

$$\pi_n = \frac{\lambda_{n-1}\lambda_{n-2}\cdots\lambda_0}{\mu_n\mu_{n-1}\cdots\mu_1}\pi_0$$

M/M/1 Queue

Queue with possion arrival and service times. Arrivals with rate λ and services with rate μ.

- Solving

$$Q = \begin{bmatrix} -\lambda & \lambda & 0 & \cdots \\ \mu & -(\mu+\lambda) & \lambda & \cdots \\ 0 & \mu & -(\mu+\lambda) & \cdots \\ 0 & 0 & \mu & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{bmatrix}$$

- Finding steady-state distribution π by equations

$$\begin{cases} (\lambda+\mu)\pi_n = \lambda\pi_{n-1} + \mu\pi_{n+1} & (n \geq 1) \\ \lambda\pi_0 = \mu\pi_1 \end{cases}$$

- Solution: Let $\rho = \lambda/\mu$, then $\pi_n = \rho^n\pi_0$.
- Normalizing: $\pi_0 = 1-\rho$, i.e., $\pi_n = \rho^n(1-\rho)$.

M/M/1: Expected Queue sizes

- Finding average number of elements in the queue:

$$\sum_{n=0}^{\infty} n\pi_n = \sum_{n=0}^{\infty} n\rho^n(1-\rho) =$$

$$(1-\rho)\sum_{n=0}^{\infty} n\rho^n = \rho(1-\rho)\sum_{n=1}^{\infty} n\rho^{n-1} =$$

$$\rho(1-\rho)\frac{d[\sum_{n=1}^{\infty}\rho^n]}{d\rho} = \rho(1-\rho)\frac{d[1/(1-\rho)]}{d\rho} =$$

$$\rho(1-\rho)(1-\rho)^{-2} = \frac{\rho}{1-\rho} = \frac{\lambda}{\mu-\lambda}$$