Performance Analysis, Autumn 2010

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Stochastic Processes

- A *Stochastic Process* is a time-dependent random variable.
- Formally a mapping $X : \Omega \mapsto T \mapsto \mathcal{R}$
- Intuitively: "outcomes are plots of functions of time"
- if $\omega \in \Omega$ then $X(\omega)$ is a function from T to $\mathcal{R}$
- if $t \in T$ then $X(\cdot)(t)$, denoted $X_t(\cdot)$ is a random variable.
- time domain T and state-space $\subseteq \mathcal{R}$ can both be either continuous or discrete.

Markov Processes

A stochastic process is a **Markov process** if

for any $t_1 < t_2 < \ldots < t_n$ *and* $x_1, x_2, \ldots, x_n$*:*

$$P[X_{t_n} \leq x_n | X_{t_1} \leq x_1 \wedge X_{t_2} \leq x_2 \wedge \cdots \wedge X_{t_{n-1}} \leq x_{n-1}] = P[X_{t_n} \leq x_n | X_{t_{n-1}} \leq x_{n-1}]$$

i.e., "The future depends only on the present, not on the past"

Markov Chains

A **Markov chain** is a discrete-time and discrete-state Markov process.

- Time domain $0, 1, 2, \ldots,$
- Notation: $\pi_i^{(n)} = P(X_n = i)$
- Vector: $\pi^{(n)} = (\pi_0^{(n)} \pi_1^{(n)} \ldots)$
- Initially: $\pi_i^{(0)} = P(X_0 = i)$
- Assume MC is *time-homogeneous*, i.e.,

$$p_{ij} = P[X_{n+1} = j | X_n = i]$$

is independent of n.

Markov Processes: definition

A **Markov chain** consists of

- A set S of *states*, usually $\{0, 1, 2, 3 \ldots\}$ or $\{0, 1, 2, \ldots, n\}$,
- An *initial probability distribution* $\pi^{(0)}$ with $\sum_{i \in S} \pi_i^{(0)} = 1$
- a *transition probability matrix*

$$P = \begin{pmatrix} p_{00} & p_{01} & p_{02} & \cdots \\ p_{10} & p_{11} & p_{12} & \cdots \\ p_{20} & p_{21} & p_{22} & \cdots \\ \cdots & \cdots & \cdots & \cdots \end{pmatrix}$$

with $\sum_{j \in S} p_{ij} = 1$ for each $i \in S$.

Markov Processes: State probabilities

How to derive $\pi^{(n)}$, probability distribution at nth step?

- By law of total probabilities (partitioning into cases):

$$\pi_i^{(n+1)} = \sum_{j \in S} (\pi_j^{(n)} p_{ji})$$

- We can use vector notation:

$$\pi^{(n+1)} = \pi^{(n)} P$$

- In particular $\pi^{(n)} = \pi^{(0)} P^n$

Markov Processes: Cat and Mouse Example

- Given 5 boxes: cat in box 1, and a mouse in box 5 at time zero.
- At each time step, both cat and mouse jump to random adjacent box.
- If they end up in same box, the game ends.

Markov Processes: Cat and Mouse Example

State numbering:

0	(cat - mouse - -)
1	(cat - - - mouse)
2	(- cat - mouse -)
3	(- - cat - mouse)
4	end

Initial probability distribution

$$\pi^{(0)} = (01000)$$

Transition probability matrix

$$P = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Markov Processes: Cat and Mouse Example

$$P = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- One absorbing state (terminal SCC): state 4
- stationary distribution (00001)
- Expected length of game:
  - Let $Y_i = E[T_4 | X_0 = i]$
  - Solve

$$\begin{cases} Y_0 &= 1 + 1/2 Y_2 + 1/2 Y_4 \\ Y_1 &= 1 + Y_2 \\ Y_2 &= 1 + 1/4 Y_0 + 1/4 Y_1 + 1/4 Y_3 + 1/4 Y_4 \\ Y_3 &= 1 + 1/2 Y_2 + 1/2 Y_4 \\ Y_4 &= 0 \end{cases}$$

Markov Processes: Cat and Mouse Example

$$P = \begin{pmatrix} 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 1 & 0 & 0 \\ 1/4 & 1/4 & 0 & 1/4 & 1/4 \\ 0 & 0 & 1/2 & 0 & 1/2 \\ 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

- One absorbing state (terminal SCC): state 4
- stationary distribution (00001)
- Expected length of game:
  - Let $Y_i = E[T_4 | X_0 = i]$
  - In general, Restrict P and Y to $\{0, 1, 2, 3\}$.
  - let $Y^T = (Y_0 \ Y_1 \ Y_2 \ Y_3)$:

$$Y = \mathbf{1} + PY$$

Markov Processes: Cat and Mouse Example

Solving for expected length of game.

$$\begin{cases} Y_0 &= 1 + 1/2 Y_2 + 1/2 Y_4 \\ Y_1 &= 1 + Y_2 \\ Y_2 &= 1 + 1/4 Y_0 + 1/4 Y_1 + 1/4 Y_3 + 1/4 Y_4 \\ Y_3 &= 1 + 1/2 Y_2 + 1/2 Y_4 \\ Y_4 &= 0 \end{cases}$$

gives

$$\begin{aligned} Y_0 &= 2.75 \\ Y_1 &= 4.5 \\ Y_2 &= 3.5 \\ Y_3 &= 2.75 \\ Y_4 &= 0 \end{aligned}$$

Classification of MCs:

Consider a MC $\langle S, \pi^{(0)}, P \rangle$.

- Consider the directed graph $\mathcal{G}$ on S of non-zero transition probabilities
- $\mathcal{G}$ can be seen as a DAG of strongly connected components (SCCs).
- The MC is *irreducible* it $\mathcal{G}$ is only one SCC.
- The *period* of MC is the gcd of the length of all cycles in $\mathcal{G}$
- An MC with period 1 is *aperiodic*

Classification of states:

States in an MC can be of three types.

- Define $T_i = \min\{n > 0 \ : \ X_n = i\}$ (number of steps to reach i).
- state i is *transient* if $P[T_i < \infty | X_0 = i] < 1$,
- state i is *recurrent* if $P[T_i < \infty | X_0 = i] = 1$,
  - state i is *positive recurrent* if $E[T_i | X_0 = i] < \infty$,
  - state i is *null recurrent* if $E[T_i | X_0 = i] = \infty$,
- Fact: in an irreducible MC, all states are of the same type (either transient, null recurrent, or positive recurrent).

Stationary Distributions:

- If MC is irreducible and aperiodic, then

$$\pi_i = \lim_{n \to \infty} P[X_n = i | X_0 = j]$$

  exists, and is independent of j.
- If all states are transient or null recurrent, then $\pi_i = 0$.
- If all states are positive recurrent, then $\pi^{(n)}$ converges to a *stationary distribution* π, which satisfies the *balance equations*
  - $\pi = \pi P$
  - $\sum_{i \in S} \pi_i = 1$ (can be written $\pi e^T = \mathbf{1}$ where $e = \begin{pmatrix} 1 \\ 1 \\ \vdots \end{pmatrix}$)

Example: Gamblers Ruin/Random Walk

Consider a MC $\langle S, \pi^{(0)}, P \rangle$, with:

- $S = \{0, 1, 2, 3, \ldots\}$
- $\pi_i^{(0)} = 1$ if $i = A$, otherwise 0.
- $p_{i,i+1} = p$ and $p_{i,i-1} = 1 - p$ and $p_{i,j} = 0$ otherwise

MC has period 2, is irreducible.

Problems:

- What is probability of reaching $B > A$ before reaching 0?
- What is the stationary distribution?

Example: Gamblers Ruin/Random Walk

Finding probability of reaching $B > A$ before reaching 0?

- For $0 \leq i \leq B$, define Z_i as probability of reaching $B > A$ before reaching 0?
- Equations:

$$\begin{cases} Z_0 & = & 0 \\ Z_i & = & pZ_{i+1} + (1-p)Z_{i-1} \text{for } 0 < i < B \\ Z_B & = & 1 \end{cases}$$

- Solve the equations: The equation

$$Z_i = pZ_{i+1} + (1-p)Z_{i-1} \text{for } 0 < i < B$$

is a homogeneous difference equation, which can be solved by assuming a solution of form

$$Z_i = c_1\alpha_1^i + c_2\alpha_2^i$$

Example: Gamblers Ruin/Random Walk

Finding stationary distribution?

- The balance equations become:

$$\begin{cases} \pi_0 & = & (1-p)\pi_1 \\ \pi_i & = & p\pi_{i-1} + (1-p)\pi_{i+1} \text{for } 0 < i \\ \sum_{i \in S} \pi_i & = & 1 \end{cases}$$