Performance Analysis 2010, Homework 5

Problem 1 A single switchboard is used to direct calls coming to a doctor's office. The calls arrive in Poisson process at a rate of 15 per hour. Call holding times can be assumed to be exponential with a mean of 2 minutes. What is the probability that the calls will not have to wait for more than 2 minutes before getting to the receptionist. Suppose it is decided to establish an upper limit K for the number of calls waiting such that the waiting time will be less than 2 minutes with a 90% probability. Determine K.

Problem 2 Consider a cyclic network with M nodes and routing probabilities is as follows:

$$r_{ij} = \begin{cases} 1 & (j = i+1, 1 \leq i \leq M-1) \\ 1 & i = M, j = 1 \\ 0 & elsewhere \end{cases}$$

The service time at each node is exponentially distributed with the same rate μ. What is the mean response time and queue length for each node when there are K jobs in the system?

Problem 3 We have an open queueing network with 3 nodes. The routing probability matrix is:

$$\begin{bmatrix} 0 & 1-p & p \\ p & 0 & 0 \\ 0 & p & 0 \end{bmatrix}$$

We also have $p_{01} = 1$, $p_{20} = 1-p$ and $p_{30} = 1-p$. The arrival rate is $\lambda_{01} = \lambda$. What is the mean response time of this network?

Problem 4 Two special-purpose machines are desired to be operational at all times. We call the operating node of this network node 1. The machines break down according to an exponential distribution with mean failure rate λ. Upon breakdown, a machine has a probability r_{12} that it can be repaired locally (node 2) by a single repairperson who works according to an exponential distribution with parameter μ_2. With probability $1-r_{12}$ the machine must be repaired by a specialist (node 3), who also works according to an exponential distribution, but with mean rate μ_3. Further, after completing local service at node 2, there is a probability r_{23} that a machine will also require the special service (the probability of returning to operation from node 2 is then $1 - r_{23}$). After the special service (node 3), the unit always returns to operation ($r_{31} = 1$). What is the probability that both machines are operating?

Problem 5 Cary Meback, the president of a large Virginia supermarket chain, is experimenting with a new store design and has remodeled one of his stores as follows. Instead of the usual checkout-counter design, the store has been remodeled to include a checkout "lounge". As customers complete their shopping, they enter the lounge with their carts and, if all checkers are busy, receive a number. They then park their carts and take a seat. When a checker is free, the next number is called and the customer with that number enters the available checkout counter. The store has been enlarged so that for practical purposes, there is a no limit on either the number of shoppers that can be in the food aisles or the number that can wait in the lounge, even during peak periods. The management estimates that during peak hours customers arrive according to a Poisson process at a mean rate of $40/h$ and it takes a customer, on average 3/4 hours to fill a shopping cart, the filling times being approximately exponentially distributed. Furthermore, the checkout times are also approximately exponentially distributed with a mean of 4 min, regardless of the particular checkout counter (during peak periods each counter has a cashier and bagger, hence the low mean checkout time). Medback wishes to know the following:

- What is the minimum number of checkout counters required in operation during peak periods?
- If it is decided to add two more than the minimum number of counters required in operation, what is the average waiting time in the lounge? How many people, on average, will be in the lounge? How many people, on average, will be in the entire supermarket?

Problem 6 How large can the utilization in an $M/M/1$ queue be so that the expected response time is at most n times the expected service time?

Problem 7 Show that an $M/M/1$ is always better with respect to L (number of jobs in the queue) than an $M/M/2$ with the same ρ.

Problem 8 Consider a closed queueing network with $K = 3$ nodes, $N = 3$ jobs, and the service discipline FCFS. The first node has $m_1 = 2$ servers and the other two nodes have one server each. The routing probabilities are given by

$$\begin{array}{lll} r_{11} = 0.6 & r_{21} = 0.5 & r_{13} = 0.4 \\ r_{12} = 0.3 & r_{22} = 0.0 & r_{32} = 0.6 \\ r_{13} = 0.1 & r_{23} = 0.5 & r_{33} = 0.0 \end{array}$$

The service times are exponentially distributed with the rates $\mu_1 = 0.4 sec^{-1}$, $\mu_2 = 0.6 sec^{-1}$, $\mu_3 = 0.3 sec^{-1}$. What is the steady-state probability that there are two jobs at node 2?