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Graduate course on Stochastic Dynamic Systems – Problems for Homework Assignement 2

3.1 Consider a system given by

$$y(t) = \frac{q+0.5}{q^2-1.5q+0.7}u(t) + \frac{q+0.7}{q-0.8}\varepsilon(t) \; ,$$

where $\varepsilon(t)$ is white noise of zero mean and unit variance. Convert it into an ARMAX model. Also represent it in a state space model of the form

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + v(t) \; , \\ y(t) &= Hx(t) + Du(t) + e(t) \; , \end{aligned}$$

where $v(t)$ and $e(t)$ are white noise sequences with zero mean and covariance matrices

$$\mathbf{E} \begin{pmatrix} v(t) \\ e(t) \end{pmatrix} \begin{pmatrix} v^T(t) & e^T(t) \end{pmatrix} = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} \; .$$

Give R_1, R_{12} and R_2. Can the system be represented in such a state space form with $R_{12} = 0$?

3.3 Examine the bounds in (3.37), (3.38), (3.39) and (3.40) by some numerical examples. Are the bounds crude or sharp?

The equations (3.37) - (3.40) are the following ones:

The maximal and minimal eigenvalues of $R(m)$ satisfy

$$\begin{aligned}
\lambda_{\max}(R(m+1)) &\geq \lambda_{\max}(R(m)) \ , & (1)\\
\lambda_{\min}(R(m+1)) &\leq \lambda_{\min}(R(m)) \ , & (2)\\
\lim_{m\to\infty} \lambda_{\max}(R(m)) &\geq \sup_{\omega} \sup_{u} \frac{u^*\phi(\mathrm{e}^{\mathrm{i}\omega})u}{u^*u} = \sup_{\omega} \lambda_{\max}[\phi(\mathrm{e}^{\mathrm{i}\omega})] \ , & (3)\\
\lim_{m\to\infty} \lambda_{\min}(R(m)) &\leq \inf_{\omega} \inf_{u} \frac{u^*\phi(\mathrm{e}^{\mathrm{i}\omega})u}{u^*u} = \inf_{\omega} \lambda_{\min}[\phi(\mathrm{e}^{\mathrm{i}\omega})] \ . & (4)
\end{aligned}$$

3.4 Consider the system

$$\begin{aligned}
x(t+1) &= Fx(t) + v(t) \ ,\\
y(t) &= Hx(t) + e(t) \ ,
\end{aligned}$$

where $v(t)$ and $e(t)$ are jointly Gaussian white noise sequences with zero mean and covariance matrices

$$\begin{aligned}
\mathbf{E}\, v(t)v^T(t) &= R_1 \ ,\\
\mathbf{E}\, v(t)e^T(t) &= R_{12} \ ,\\
\mathbf{E}\, e(t)e^T(t) &= R_2 \ .
\end{aligned}$$

The initial value $x(t_0)$ is assumed to be Gaussian distributed, $x(t_0) \sim \mathbf{N}(m_0, R_0)$, and independent of the noise sources.

(a) Derive the conditional pdfs $p(x(t+1)|x(t))$ and $p(x(t+1)|x(t), y(t))$. When do they coincide?

Hint. If A, D and the matrix below are invertible:

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}^{-1} = \begin{pmatrix} A^{-1} & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} -A^{-1}B \\ I \end{pmatrix} (D - CA^{-1}B)^{-1} \times (-CA^{-1} \quad I) \ .$$

holds.

(b) In case the two conditional pdfs differ, show that one can use

$$\overline{x}(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

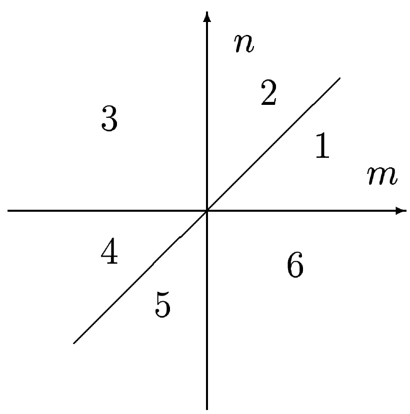
as a state vector. Show that it satisfies

$$\overline{x}(t+1) = \overline{F}\,\overline{x}(t) + \overline{v}(t)$$

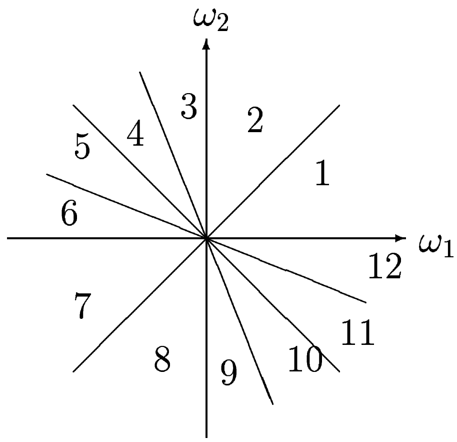
with $\overline{v}(t)$ white noise, and give explicit expressions for $\overline{F}$ and $\overline{R}_1 = \mathbf{E}\,\overline{v}(t)\overline{v}^T(t)$.

3.6 Examine various "symmetries" of the third-order moment sequence and the bispectrum.

(a) Show that if a third order moment sequence is known in any of the six sectors below, it can easily be determined for all arguments:



(b) Show that if the bispectrum is known in any of the 12 sectors below, then it can easily be determined for all arguments:



The bispectrum can be considered for arguments $z_k = \mathrm{e}^{\mathrm{i}\omega_\mathrm{k}}, \;\; \mathrm{k} = 1, 2$. This is sufficient since $B(z_1, z_2)$ is analytic. Note that the angles of the sectors are not 22.5 degrees.