Stochastic dynamic systems
Chapter 9

Nonlinear filtering

- Extended Kalman filtering
- Gaussian sum estimators
- Quantized measurements
- Median filters

Extended Kalman filtering (EKF)

Nonlinear model

$$\begin{aligned} x(t+1) &= f(x(t)) + v(t) \\ y(t) &= h(x(t)) + e(t) \end{aligned}$$

Principle: Linearize the model, and apply the Kalman filter.

Linearization

$$\begin{aligned} F(t) &= \left. \frac{\partial f(t,x)}{\partial x} \right|_{x=\hat{x}(t|t)} \\ G(t) &= g(t,x)|_{x=\hat{x}(t|t)} \\ H(t) &= \left. \frac{\partial h(t,x)}{\partial x} \right|_{x=\hat{x}(t|t-1)} \end{aligned}$$

EKF, cont'd

Linearized model

$$\begin{aligned} x(t+1) &= F(t)x(t) + G(t)v(t) + \tilde{u}(t) \\ y(t) &= H(t)x(t) + e(t) + w(t) \\ \tilde{u}(t) &= f(t, \hat{x}(t|t)) - F(t)\hat{x}(t|t) \\ w(t) &= h(t, \hat{x}(t|t-1)) - H(t)\hat{x}(t|t-1) \end{aligned}$$

Extended Kalman filtering

Algorithm

$$
\begin{aligned}
H(t) &= \left.\frac{\partial h(t,x)}{\partial x}\right|_{x=\hat{x}(t|t-1)} \\
K(t) &= P(t|t-1)H^T(t) \\
&\quad \times [H(t)P(t|t-1)H^T(t) + R_2(t)]^{-1} \\
\hat{x}(t|t) &= \hat{x}(t|t-1) + \\
&\quad + K(t)[y(t) - h(t, \hat{x}(t|t-1))] \\
P(t|t) &= P(t|t-1) - K(t)H(t)P(t|t-1) \\
\hat{x}(t+1|t) &= f(t, \hat{x}(t|t)) \\
F(t) &= \left.\frac{\partial f(t,x)}{\partial x}\right|_{x=\hat{x}(t|t)} \\
G(t) &= g(t,x)|_{x=\hat{x}(t|t)} \\
P(t+1|t) &= F(t)P(t|t)F^T(t) + G(t)R_1(t)G^T(t)
\end{aligned}
$$

Extended Kalman filtering - variants

1. Modify the linearization point. ($\hat{x}(t|t)$ can be updated).

2. A second order filter. Example

$$
\begin{aligned}
f(t, x(t)) &\approx f(t, \hat{x}(t|t)) \\
&\quad + F(t)(x(t) - \hat{x}(t|t)) + \bar{f}(t) \\
g(t, x(t)) &\approx G(t) \\
h(t, x(t)) &\approx h(t, \hat{x}(t|t-1)) \\
&\quad + H(t)(x(t) - \hat{x}(t|t-1)) + \bar{h}(t)
\end{aligned}
$$

EKF variants, cont'd

$$
\begin{aligned}
\tilde{x}(t|t) &= x(t) - \hat{x}(t|t) \\
\tilde{x}(t|t-1) &= x(t) - \hat{x}(t|t-1) \\
(\bar{f}(t))_k &= \frac{1}{2}\tilde{x}(t|t)^T \left.\frac{\partial^2 f_k}{\partial x^2}\right|_{x=\hat{x}(t|t)} \tilde{x}(t|t) \\
&\approx \frac{1}{2}\mathrm{tr}\left[P(t|t) \left.\frac{\partial^2 f_k}{\partial x^2}\right|_{x=\hat{x}(t|t)}\right] \\
(\bar{h}(t))_k &= \frac{1}{2}\tilde{x}(t|t-1)^T \left.\frac{\partial^2 h_k}{\partial x^2}\right|_{x=\hat{x}(t|t-1)} \tilde{x}(t|t-1) \\
&\approx \frac{1}{2}\mathrm{tr}\left[P(t|t-1) \left.\frac{\partial^2 h_k}{\partial x^2}\right|_{x=\hat{x}(t|t-1)}\right]
\end{aligned}
$$

Nonlinear estimators – comparative example

$$
\begin{aligned}
z &= x^2 \\
y &= x + e, \qquad x \perp e
\end{aligned}
$$

x uniform over $(-a, a)$
e uniform over $(-b, b)$

Estimate x and z from y

1. EKF

2. Second order EKF

3. Mean square optimal estimates
   $\hat{x} = E[x|y] = \int xp(x|y)dx$
   $\hat{z} = E[z|y] = \int z^2 p(z|y)dz$

Nonlinear estimators – comparative example, cont'd

Dynamics:

$$
\begin{aligned}
x(t+1) &= x^2(t) \\
y(t) &= x(t) + e(t)
\end{aligned}
$$

$[t = 0 : z = x(1), x = x(0), e = e(0), y = y(0)]$

Initial estimate:

$$\widehat{x}(0|-1) = 0 \qquad P(0|-1) = a^2/3$$

EKF estimate of $\widehat{x}$:

$$
\begin{aligned}
\widehat{x}(0|0) &= K(0)y(0) = \frac{a^2}{a^2+b^2}y(0) \\
P(0|0) &= \frac{a^2b^2}{3(a^2+b^2)}
\end{aligned}
$$

Nonlinear estimators – comparative example, cont'd

EKF:

$$
\begin{aligned}
\widehat{z} &= \widehat{x}(1|0) = \widehat{x}^2(0|0) \\
&= \left[\frac{a^2}{a^2+b^2}\right]^2 y^2
\end{aligned}
$$

Second order EKF: $[f(x) = x^2]$

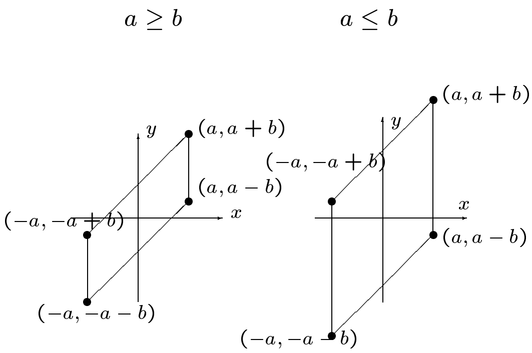
$$
\begin{aligned}
\widehat{z} &= \widehat{x}(1|0) \\
&= f(\widehat{x}(0|0)) + \frac{1}{2}f''(\widehat{x}(0|0))P(0|0) \\
&= \widehat{x}^2(0|0) + P(0|0) \\
&= \widehat{x}^2(0|0) + \frac{a^2b^2}{3(a^2+b^2)} \\
&= \left[\frac{a^2}{a^2+b^2}\right]^2 y^2 + \frac{a^2b^2}{3(a^2+b^2)}
\end{aligned}
$$

Nonlinear estimators – comparative example, cont'd

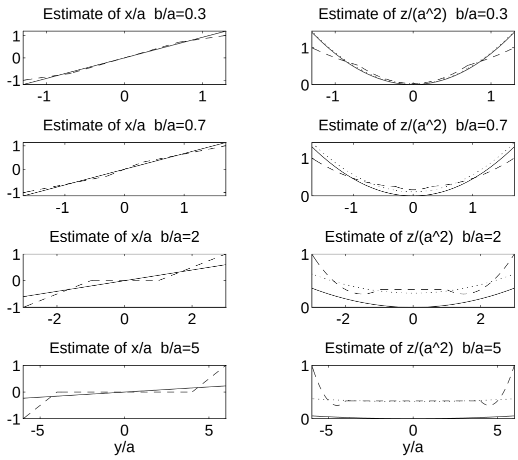
$\widehat{x} = E[x|y] = \int xp(x|y)dx$

$$p(x|y) = \frac{p(x,y)}{p(y)}$$

Joint pdf $p(x,y)$:



Nonlinear estimators – comparative example, cont'd



EKF [solid], the second order EKF [dotted], optimal estimate [dashed].

Nonlinear estimators – comparative example, cont'd

Monte Carlo simulations

$$a = 1$$

Method	b	0.3	0.7	2	5
EKF	MSE($\hat{x}$)	0.027	0.108	0.27	0.32
	MSE($\hat{z}$)	0.031	0.085	0.16	0.19
Second order EKF	MSE($\hat{x}$)	0.027	0.108	0.27	0.32
	MSE($\hat{z}$)	0.030	0.075	0.087	0.088
Optimal estimate	MSE($\hat{x}$)	0.025	0.105	0.25	0.30
	MSE($\hat{z}$)	0.024	0.064	0.081	0.085

Conclusions

- MSE($\hat{x}$): All methods similar results
- MSE($\hat{z}$): EKF inferior for large b
- Optimal estimates have smallest MSE's

Gaussian sum estimators

Idea: Approximate the pdf's

$$p(x) \sim \sum_{i=1}^{m} \alpha_i \gamma(x; m_i, P_i)$$

$$\alpha_i \geq 0, \quad \sum \alpha_i = 1$$
$\gamma(x; m_i, P_i)$ Gaussian distr

Approximations can be made arbitrarily good

Gaussian sum estimators

Nonlinear system

$$\begin{aligned} x(t+1) &= f(x(t)) + g(x(t))v(t) \\ y(t) &= h(x(t)) + e(t) \end{aligned}$$

$$Ev(t)v^T(t) = R_1, \quad Ee(t)e^T(t) = R_2$$

Approximate pdf's

$$\begin{aligned} p(x(t)|Y^{t-1}) &\sim \sum_{i=1}^{m} \alpha_i(t-1)\gamma(x(t); \overline{m}_i(t), \overline{P}_i(t)) \\ p(x(t)|Y^{t}) &\sim \sum_{i=1}^{m} \alpha_i(t)\gamma(x(t); m_i(t), P_i(t)) \end{aligned}$$

Update as an extended Kalman filter, combined with

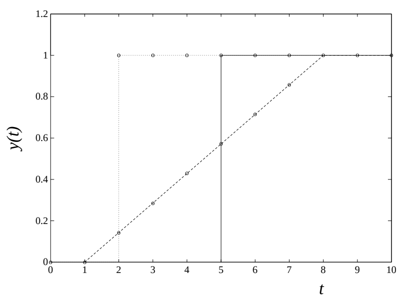
$$\alpha_i(t) = \frac{\alpha_i(t-1)\gamma(y(t); h(\overline{m}_i(t)), S_i(t))}{\sum_{i=1}^{m} \alpha_i(t-1)\gamma(y(t); h(\overline{m}_i(t)), S_i(t))}$$

Median filters

An alternative to a linear FIR filter:

$y(t)$ is the median of
$u(t),\ u(t-1),\ \ldots,\ u(t-2m)$.

Step responses



Step responses for median filter (solid) and mean filter (dashed), $n = 7$ ($m = 3$).