Stochastic dynamic systems

Chapter 7

Optimal estimation for linear polynomial system

- Optimal prediction of ARMA processes
- Optimal prediction
- Wiener filter
- Minimum variance filter
- Robustness against modelling errors

Optimal prediction of ARMA models

ARMA process

$$A(q)y(t) = C(q)e(t) \qquad Ee(t) = \lambda^2$$

$$\begin{aligned} A(q) &= q^n + a_1 q^{n-1} + \ldots + a_n \\ C(q) &= q^n + c_1 q^{n-1} + \ldots + c_n \end{aligned}$$

Predictor identity

$$z^{k-1}C(z) \equiv A(z)F(z) + L(z)$$

$$\begin{aligned} F(z) &= z^{k-1} + f_1 z^{k-2} + \ldots + f_{k-1} \\ L(z) &= \ell_o z^{n-1} + \ell_1 z^{n-2} + \ldots + \ell_{n-1} \end{aligned}$$

Optimal prediction of ARMA models, cont'd

Optimal predictor

$$\hat{y}(t+k|t) = \frac{qL(q)}{A(q)} \; \frac{A(q)}{C(q)} y(t) = \frac{qL(q)}{C(q)} y(t)$$

Prediction error

$$\begin{aligned} \tilde{y}(t+k) &= F(q)e(t+1) \\ &= e(t+k) + f_1 e(t+k-1) + \ldots \\ &\quad + f_{k-1} e(t+1) \end{aligned}$$

Optimal prediction

System (innovations form)

$$y(t) = G(q)u(t) + H(q)e(t)$$

Criterion

$$V = E[y(t+k) - f(t)]^* P[y(t+k) - f(t)]$$

Rewrite

$$\begin{aligned} H(q) &= \underbrace{[\sum_{j=0}^{k-1} h_j q^{-j}]}_{q^{-k+1}H_o(q)} + \underbrace{q^{-k}[\sum_{j=0}^{\infty} h_{k+j} q^{-j}]}_{q^{-k}H_1(q)} \\ &\triangleq q^{-k+1}H_o(q) \quad q^{-k}H_1(q) \end{aligned}$$

leading to

$$y(t+k) \triangleq \varepsilon(t+1) + z(t)$$

Optimal prediction, cont'd

Optimal predictor

$$\begin{aligned} \hat{y}(t+k|t) &= H_1(q)H^{-1}(q)y(t) \\ &\quad + H_o(q)H^{-1}(q)G(q)u(t+1) \end{aligned}$$

Prediction error

$$\tilde{y}(t+k) \triangleq y(t+k) - \hat{y}(t+k|t) = H_o(q)e(t+1)$$

Wiener filters — general aspects

Estimate $s(t)$ based on $\{y(t)\}$.

Criterion

$$V = E||s(t) - G(q)y(t)||^2$$

Rewrite

$$\tilde{s}(t) = s(t) - G(q)y(t)$$

$$V = \text{tr}\frac{1}{2\pi i}\oint \phi_{\tilde{s}}(z)\frac{dz}{z}$$

From $V' = 0$

$$0 = \text{tr}\frac{1}{2\pi i}\oint [G(z)\phi_y(z) - \phi_{sy}(z)]G_1^*(z)\frac{dz}{z}$$

for any $G_1(z)$.

Unrealizable Wiener filter

$V' = 0$ if

$$G(z) = \phi_{sy}(z)\phi_y^{-1}(z)$$

It is noncausal, hence unrealizable.

Realizable Wienerfilter

Causal and anticausal part of a filter

Let

$$G(z) = \sum_{j=-\infty}^{\infty} g_j z^{-j}$$

converge on the unit circle (Laurent series).

Causal part of $G(z)$:

$$[G(z)]_+ = \sum_{j=0}^{\infty} g_j z^{-j}$$

Anticausal part of $G(z)$:

$$[G(z)]_- = \sum_{j=-\infty}^{-1} G_j z^{-j} = G(z) - [G(z)]_+$$

has no zeros inside or on the unit circle

Realizable Wienerfilter

Innovations model

$$\begin{aligned} y(t) &= H(q)e(t) \\ Ee(t)e^*(s) &= \Lambda\delta_{t,s} \\ \\ H(0) &= I \end{aligned}$$

$$H(q), H^{-1}(q) \text{ as stable}$$

$$G(z) = [\phi_{sy}(z)\{H^*(z)\}^{-1}\Lambda^{-1}]_+ H^{-1}(z)$$

Realizable Wienerfilter – an example

ARMA process

$$\begin{aligned} A(q)y(t) &= C(q)e(t) \\ Ee^2(t) &= \lambda^2 \end{aligned}$$

Prediction problem

$$s(t) = y(t+k) \qquad k > 0$$

$$\phi_{sy}(z) = z^k \lambda^2 \frac{C(z)}{A(z)} \frac{C^*(z)}{A^*(z)}$$

Unrealizable Wiener filter

$$G(z) = z^k \phi_y(z) \phi_y^{-1}(z) = z^k$$

giving

$$\hat{s}(t) = y(t+k)$$

Realizable Wienerfilter – example (cont'd)

Realizable Wiener filter

$$G(z) = \left[z^k \frac{C(z)}{A(z)} \right]_+ \frac{A(z)}{C(z)}$$

Predictor identity

$$z^{k-1}C(z) \equiv A(z)F(z) + L(z)$$

Optimal predictor

$$\begin{aligned} G(z) &= \left[\frac{zA(z)F(z) + zL(z)}{A(z)} \right]_+ \frac{A(z)}{C(z)} \\ &= \frac{zL(z)}{A(z)} \frac{A(z)}{C(z)} = \frac{zL(z)}{C(z)} \end{aligned}$$

Computational aspects

$$f(z) = \left[\frac{G(z,z^{-1})}{D(z)F(z^{-1})}\right]_+$$

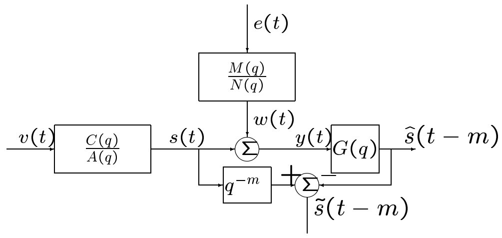
can be evaluated by solving the Diophantine equation

$$G(z,z^{-1}) \equiv z^{\alpha}F(z^{-1})Q(z) + z^{\beta}D(z)L(z^{-1})$$

Interpretation: Partial fraction decomposition!

$$\begin{aligned}\left[\frac{G(z,z^{-1})}{D(z)F(z^{-1})}\right]_+ &= \left[\frac{z^{\alpha}Q(z)}{D(z)}\right]_+ + \left[\frac{z^{\beta}L(z^{-1})}{F(z^{-1})}\right]_+ \\ &= \frac{z^{\alpha}Q(z)}{D(z)}\end{aligned}$$

Minimum variance filters



deg$A(q) = n$, deg$C(q) = n$, deg$M(q) = r$, deg$N(q)$

Spectral factorization

$$\begin{aligned}\lambda_{\varepsilon}^2 B(z)B^*(z) &= \lambda_v^2 C(z)C^*(z)N(z)N^*(z) \\ &\quad + \lambda_e^2 A(z)A^*(z)M(z)M^*(z)\end{aligned}$$

Wiener filter solution

$$G(z) = \frac{\lambda_v^2}{\lambda_{\varepsilon}^2}\left[z^{-m}\frac{C(z)C^*(z)N^*(z)}{A(z)B^*(z)}\right]_+ \frac{A(z)N(z)}{B(z)}$$

Minimum variance filters

Diophantine equation

$$\begin{aligned}z^{-m}C(z)C^*(z)N^*(z) &\equiv z^{\min(0,-m)}B^*(z)Q(z) \\ &\quad + z^{\max(0,-m)}A(z)L^*(z)\end{aligned}$$

$$\begin{aligned}dQ &= n - \min(0,-m) \\ dL &= n + r - 1 + \max(0,-m)\end{aligned}$$

Optimal filter

$$G(z) = \frac{\lambda_v^2}{\lambda_{\varepsilon}^2} \; \frac{z^{\min(0,-m)}Q(z)N(z)}{B(z)}$$

Some possible extensions

1. Multivariable processes
2. Estimation of other signals than $s(t)$
3. Prefiltering
$$V = E[P(q)\tilde{s}(t)]^2$$
(frequency weighting)

Robustness against modelling errors

Idea: Model uncertainties in the process parameters as additional noise sources. Then solve the estimation problem with the modified noise model.

Example: Prediction of an MA(1) process, c uncertain

$$\begin{aligned} y(t) &= [1 + (c + \tilde{c})q^{-1}]v(t) \\ Ev^2(t) &= \lambda^2 \end{aligned}$$

Parameter uncertainty: $\tilde{c}$ random variable, zero mean, variance σ^2, independent of $\{v(t)\}$.

Example, cont'd

Second order properties of $y(t)$:

$$\begin{aligned} Ey^2(t) &= \lambda^2 + \lambda^2(c^2 + \sigma^2) \\ Ey(t)y(t-1) &= c\lambda^2 \\ Ey(t)y(t-\tau) &= 0, \quad \tau \geq 2 \end{aligned}$$

This coincides with $r(\tau)$ of another MA(1) process

$$y(t) = [1 + \bar{c}q^{-1}]\varepsilon(t)$$

Example, cont'd

Optimal prediction

$$\hat{y}(t+1|t) = \frac{\bar{c}}{1 + \bar{c}q^{-1}} y(t)$$

Parameter $\bar{c}$ from spectral factorization:

$$\begin{aligned} \lambda^2(1 + c^2 + \sigma^2) &= \lambda_\varepsilon^2(1 + \bar{c}^2) \\ \lambda^2 c &= \lambda_\varepsilon \bar{c} \end{aligned}$$