Stochastic dynamic systems
Chapter 6

Optimal estimation for linear state space systems

- Linear mean square one-step prediction
- Conditional mean
- Optimal filtering and prediction
- Optimal smoothing
- ML interpretation
- The stationary case
- Algorithms

Optimal state estimation

State space model

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

$$E \begin{pmatrix} v(t) \\ e(t) \end{pmatrix} (v^*(s) \quad e^*(s)) = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} \delta_{t,s}$$

Estimate $x(t)$ based on

$Y^s = \{y(s), u(s), y(s-1), u(s-1), \dots, y(t_o), u(t_o)\}$

Three cases

- If $t > s$, prediction
- If $t = s$ filtering
- If $t < s$ smoothing

Optimal state estimation, cont'd

Minimize

$$P(t|s) = E[x(t) - \hat{x}(t|s)][x(t) - \hat{x}(t|s)]^*$$

Approaches

1. LMS estimate
2. Conditional mean

Optimal one–step prediction

Initial state estimate $\hat{x}(t_o|t_o-1)$

$$R(t_o) = E[x(t_o) - \hat{x}(t_o|t_o-1)][x(t_o) - \hat{x}(t_o|t_o-1)]^*$$

Output innovations

$$\begin{aligned} \tilde{y}(t) &= y(t) - \hat{y}(t|t-1) \\ &= y(t) - H\hat{x}(t|t-1) \end{aligned}$$

Optimal predictor

$$\hat{x}(t+1|t) = F\hat{x}(t|t-1) + Gu(t) + K(t)\tilde{y}(t)$$

$$\begin{aligned} K(t) &= [FP(t|t-1)H^* + R_{12}] \\ &\quad \times [HP(t|t-1)H^* + R_2]^{-1} \end{aligned}$$

Riccati equation

$$\begin{aligned} P(t+1|t) &= FP(t|t-1)F^* + R_1 \\ &\quad -K(t)[HP(t|t-1)F^* + R_{21}] \end{aligned}$$

The conditional mean

State space model with Gaussian disturbances

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

$$E \begin{pmatrix} v(t) \\ e(t) \end{pmatrix} (v^*(s) e^*(s)) = \begin{pmatrix} R_1 & 0 \\ 0 & R_2 \end{pmatrix} \delta_{t,s}$$

Conditional pdf

$$\begin{aligned} p(x(t)|Y^t) &\sim \mathbf{N}(\hat{x}(t|t), P(t|t)) \\ p(x(t+1)|Y^t) &\sim \mathbf{N}(\hat{x}(t+1|t), P(t+1|t)) \end{aligned}$$

Conditional means $\hat{x}(t|t)$ and $\hat{x}(t+1|t)$ and their covariance matrices $P(t|t)$, $P(t+1|t)$ are given by the Kalman filter.

Optimal filter

Optimal filter

$$\hat{x}(t|t) = \hat{x}(t|t-1) + K_f(t)[y(t) - H\hat{x}(t|t-1)]$$

$$\begin{aligned} K_f(t) &= P(t|t-1)H^*\Lambda^{-1}(t) \\ \Lambda(t) &= HP(t|t-1)H^* + R_2 \end{aligned}$$

Relations

$$K_p(t) = FK_f(t)$$

$$\hat{x}(t+1|t) = F\hat{x}(t|t) + Gu(t)$$

$$\Lambda = E\tilde{y}(t)\tilde{y}^T(t)$$

Optimal prediction

$$\begin{aligned} \hat{x}(t+k|t) &= F\hat{x}(t+k-1|t) + Gu(t+k-1) \\ &= F^k\hat{x}(t|t) + \sum_{s=t}^{t+k-1} F^{t+k-1-s}Gu(s) \end{aligned}$$

$$P(t+k|t) = F^k P(t|t) F^{T^k} + \sum_{s=0}^{k-1} F^s R_1 F^{T^s}$$

Optimal smoothing

Consider augmented system

$$\begin{aligned} z(t) &\triangleq \begin{pmatrix} x(t) \\ x(t_o) \end{pmatrix} \qquad t \geq t_o \\ z(t+1) &= \begin{pmatrix} F & 0 \\ 0 & I \end{pmatrix} z(t) + \begin{pmatrix} v(t) \\ 0 \end{pmatrix} \\ y(t) &= (H \quad 0) z(t) + e(t) \\ z(t_o) &= \begin{pmatrix} x(t_o) \\ x(t_o) \end{pmatrix} \end{aligned}$$

Apply standard Kalman filter.

Optimal smoothing

$$\hat{x}(t_o|t) = \hat{x}(t_o|t_o) + \sum_{s=t_o+1}^{t} K_2(s)[y(s) - H\hat{x}(s|s-1)]$$

$$K_2(t) = P_{12}^*(t|t-1)H^*[HP(t|t-1)H^* + R_2]^{-1}$$

$$\begin{aligned} P_{12}(t+1|t) &= [F - K(t)H]P_{12}(t|t-1) \\ P_{12}(t_o|t_o-1) &= P(t_o|t_o-1) \end{aligned}$$

Stationary case

Factorize $R_1 = BB^T$

Assume

1. (F, B) controllable
2. $R_2 > 0$

Then $P(t) \to P$ as $t - t_o \to \infty$

$$P = FPF^T + R_1 - FPH^T(HPH^T + R_2)^{-1}HPF^T$$

Cf Spectral factorization!

$F - KH$ all eigenvalues strictly inside unit circle

Solving the Riccati equation

General form

$$\begin{aligned} P(t+1) &= FP(t)F^T + R_1 - [FP(t)H^T + R_{12}] \\ &\quad \times [HP(t)H^T + R_2]^{-1}[HP(t)F^T + R_{21}] \end{aligned}$$

Reduce to form with $R_{12} = 0$.

Set

$$\begin{aligned} \overline{F} &\triangleq F - R_{12}R_2^{-1}H \\ \overline{R}_1 &\triangleq R_1 - R_{12}R_2^{-1}R_{21} \end{aligned}$$

This leads to

$$\begin{aligned} P(t+1) &= \overline{F}P(t)\overline{F}^T + \overline{R}_1 - [\overline{F}P(t)H^T] \\ &\quad \times [HP(t)H^T + R_2]^{-1}[HP(t)\overline{F}^T] \end{aligned}$$

The Euler system

Riccati equation

$$\begin{aligned} P(t+1) &= FP(t)F^T + R_1 - FP(t)H^T \\ &\quad \times [HP(t)H^T + R_2]^{-1} HP(t)F^T \\ P(0) &= P_0 \end{aligned}$$

Euler matrix

$$\phi \triangleq \begin{pmatrix} F^{-T} & F^{-T}H^T R_2^{-1} H \\ R_1 F^{-T} & F + R_1 F^{-T} H^T R_2^{-1} H \end{pmatrix}$$

Euler system

$$\begin{pmatrix} Y(t+1) \\ Z(t+1) \end{pmatrix} = \phi \begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix}, \; Y(0) = I, \;\; Z(0) = P_0$$

Result

$$P(t) = Z(t)Y^{-1}(t)$$

The Euler system

Result: If $\lambda \neq 0$ is eigenvalue of ϕ, so is λ^{-1}.

Solution to the Euler system:

$$\begin{pmatrix} Y(t) \\ Z(t) \end{pmatrix} = \phi^t \begin{pmatrix} Y(0) \\ Z(0) \end{pmatrix} = \phi^t \begin{pmatrix} I \\ P_0 \end{pmatrix}$$

Solution to the algebraic Riccati equation. Ordered eigenvectors of ϕ:

$$\begin{pmatrix} U_{11} & U_{12} \\ U_{21} & U_{22} \end{pmatrix}$$

Sought solution to ARE is

$$P = U_{21} U_{11}^{-1}$$