Stochastic dynamic systems

Chapter 5

Optimal estimation

- The conditional mean
- Best linear estimate
- ML estimation

Optimal estimation

Problem: Estimate x, given y

"Optimal" estimate: $\hat{x} = E[x|y]$

Underlying assumptions:

1. Symmetric criterion
2. Symmetric cond pdf

Best linear estimate

Given

$$Ex = m_x, \quad Ey = m_y$$

$$E \begin{pmatrix} x - m_x \\ y - m_y \end{pmatrix} (x^* - m_x^* \quad y^* - m_y^*)$$

$$= \begin{pmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{pmatrix}$$

Seek estimate of the form

$$\hat{x} = Ay + b$$

and minimize

$$J = E(x - \hat{x})(x - \hat{x})^*$$

Solution

$$\hat{x}_{\mathsf{LMS}} = m_x + R_{xy} R_y^{-1} (y - m_y)$$

$$\min J = R_x - R_{xy} R_y^{-1} R_{yx}$$

Best linear estimate – interpretations

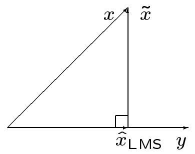
1. Conditional mean and cov in Gaussian case

2. Geometrical interpretation

$$E[x - \hat{x}_{\text{LMS}}]y^* = 0$$

Inner product

$$< x|y >= E(x - m_x)^T(y - m_y)$$



Dynamics of cond pdf

System

$$\begin{aligned} x(t+1) &= f(x(t), v(t)) \\ y(t) &= h(x(t), e(t)) \end{aligned}$$

$v(t)$ and $e(t)$ independent, white noise

To find $\hat{x}(t) = E[x(t)|Y^t]$ we need $p(x|Y^t)$.

$$p(x(t+1)|Y^{t+1}) =$$

$$= \frac{\int p(y(t+1)|x(t+1))p(x(t+1)|x(t))p(x(t)|Y^t)dx(t)}{\int\int p(y(t+1)|x(t+1))p(x(t+1)|x(t))p(x(t)|Y^t)dx(t)dx(t+1)}$$

- $p(y(t+1)|x(t+1))$ from $h(\cdot,\cdot)$

- $p(x(t+1)|x(t))$ from $f(\cdot,\cdot)$

Maximum likelihood estimation

Likelihood function

$$L(\theta) = p(y|\theta)$$

ML estimate

$$\hat{\theta}_{ML} = \text{arg } \min_{\theta} L(\theta)$$

Example: Linear system with Gaussian disturbances

The ML estimate of the states is the LMS estimate!