## Stochastic dynamic systems
## Chapter 4





# Analysis

- Linear filtering
- Spectral factorization
- Continuous-time models
- Sampling
- The positive real part of the spectrum
- Filtering and bispectrum
- Computational algorithms





# Linear filtering

$$y(t) = G(q)u(t) = \sum_{k=0}^{\infty} g_k u(t-k)$$

$u(t)$ stationary, $G(q)$ asymptotically stable

Then $y(t)$ stationary.

$$\begin{aligned} m_y &= G(1)m_u \\ \phi_y(z) &= G(z)\phi_u(z)G^*(z^{-*}) \\ \phi_{yu}(z) &= G(z)\phi_u(z) \end{aligned}$$

### Parseval's relation

$u(t)$ uncorrelated sequence, $\text{cov}(u(t)) = \Lambda \Rightarrow$

$$Ey(t)y^*(t) = \sum_{k=0}^{\infty} g_k \Lambda g_k^* = \frac{1}{2\pi i}\oint G(z)\Lambda G^*(z)\frac{dz}{z}$$





# Linear filtering

State space model

$$\begin{aligned} x(t+1) &= Fx(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

$$E\begin{pmatrix} v(t) \\ e(t) \end{pmatrix}\begin{pmatrix} v^*(s) & e^*(s) \end{pmatrix} = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix}\delta_{t,s}$$

$x(t_o)$ independent of $v(t)$ and $e(t)$ for all $t$, $Ex(t_o) = m_o$, $\text{cov}(x(t_o)) = R_o$

$$\begin{aligned} m_x(t) &= Ex(t) \\ P(t) = \text{cov}[x(t)] &= E[x(t) - m(t)][x(t) - m(t)]^* \end{aligned}$$

## Linear filtering, cont'd

$$\begin{aligned} m_x(t+1) &= F m_x(t) = F^{t+1-t_o} m_o \\ R_x(t+\tau, t) &= F^{\tau} P(t) \qquad \tau \geq 0 \\ P(t+1) &= F P(t) F^* + R_1 \qquad P(t_o) = R_o \end{aligned}$$

Lyapunov equation

$$\begin{aligned} m_y(t) &= H m_x(t) \\ R_y(t,t) &= H P(t) H^* + R_2 \\ R_y(t+\tau, t) &= H F^{\tau-1}[F P(t) H^* + R_{12}] \qquad \tau > 0 \end{aligned}$$





## Linear filtering — stationary case

For asymptotically stable systems:

$P(t)$ converges to solution of the Lyapunov equation

$$P = FPF^* + R_1$$

as $t - t_o \to \infty$.

Unique solution $\exists$, is Hermitian and nonnegative definite.

### Spectrum

$$\begin{aligned} \phi_y(z) &= H(zI-F)^{-1} R_1 (z^{-*}I - F^*)^{-1} H^* + R_2 \\ &\quad + H(zI-F)^{-1} R_{12} + R_{21}(z^{-*}I - F^*)^{-1} H \end{aligned}$$





## Yule-Walker equation

1. Autoregressive process

   $$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) = e(t)$$

   Multiply by $y(t-\tau)$, $\tau \geq 0$. Take expectation

   $$r(\tau) + a_1 r(\tau-1) + \ldots + a_n r(\tau-n) = \begin{cases} 0 & \tau > 0 \\ \lambda^2 & \tau = 0 \end{cases}$$





## Yule-Walker equation, cont'd

1. Full ARMA process

   $$\begin{aligned} &y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) \\ &\quad = e(t) + c_1 e(t-1) + \ldots + c_m e(t-m) \end{aligned}$$

   Multiply by $y(t-\tau)$ to get difference equation in $r_y(\tau) = r(\tau)$ and $r_{ey}(\tau)$.

   Multiply by $e(t-\tau)$ to get difference equation in $r_{ey}(\tau)$ and $r_e(\tau)$.

   Eliminate $r_{ey}(\tau)$.

   $$r(\tau) + a_1 r(\tau-1) + \ldots + a_n r(\tau-n) = 0, \quad \tau > m$$

## Spectral factorization

Given

$$\phi(e^{i\omega}) = \frac{\sum_{|k|\le m} \beta_k e^{ik\omega}}{\sum_{|k|\le n} \alpha_k e^{ik\omega}}$$

$\exists\ \lambda^2 > 0$ and two polynomials

$$\begin{aligned} A(z) &= z^n + a_1 z^{n-1} + \ldots + a_n \\ C(z) &= z^m + c_1 z^{m-1} + \ldots + c_m \end{aligned}$$

i) $A(z)$ has all zeros inside the unit circle.

ii) $C(z)$ has all zeros inside or on the unit circle.

iii)

$$\phi(e^{i\omega}) = \lambda^2\ \frac{C(e^{i\omega})}{A(e^{i\omega})}\ \overline{\frac{C(e^{i\omega})}{A(e^{i\omega})}} = \lambda^2 \left|\frac{C(e^{i\omega})}{A(e^{i\omega})}\right|^2$$

In case $\phi(\omega) > 0$ for all $\omega$, $C(z)$ will have no zeros on the circle.

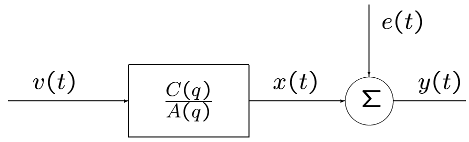


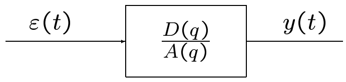


## Spectral factorization

Interpretation

Regard the process as generated by filtering white noise




$$\begin{aligned} \phi(z) &= \lambda_e^2 + \lambda_v^2 \frac{CC^*}{AA^*} = \lambda_\varepsilon^2 \frac{DD^*}{AA^*} \\ \lambda_\varepsilon^2 DD^* &\equiv \lambda_v^2 CC^* + \lambda_e^2 AA^* \end{aligned}$$





## Spectral factorization — state space models

Asymptotically stable system

$$\begin{aligned} x(t+1) &= Fx(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

$$E \begin{pmatrix} v(t) \\ e(t) \end{pmatrix} (v^*(s)\quad e^*(s)) = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix} \delta_{t,s}$$

Then

$$\phi_y(z) = H(z)\Lambda H^*(z^{-*})$$

i) $H(\infty) = I$

ii) $H(q)$ asymptotically stable

iii) $H^{-1}(q)$ (asymptotically) stable





## Spectral factorization, cont'd

How to find $H(q)$ and $\Lambda$?

Answer:

$$\begin{aligned} H(q) &= I + H(qI - F)^{-1} K \\ \Lambda &= HPH^* + R_2 \\ K &= (FPH^* + R_{12})(HPH^* + R_2)^{-1} \\ P &= FPF^* + R_1 - (FPH^* + R_{12}) \\ &\quad \times (HPH^* + R_2)^{-1}(HPF^* + R_{21}) \end{aligned}$$

## Spectral factorization

Innovations representation

$$y(t) = H(q)\tilde{y}(t) \qquad E\tilde{y}(t)\tilde{y}^*(s) = \Lambda\delta_{t,s}$$

$$\begin{aligned} x_o(t+1) &= Fx_o(t) + K\tilde{y}(t) \\ y(t) &= Hx_o(t) + \tilde{y}(t) \end{aligned}$$

$$H(q) = I + H(qI - F)^{-1}K \qquad \Lambda = HPH^* + R_2$$





## Continuous time models

$$\phi(s) = \int_{-\infty}^{\infty} r(\tau)e^{-s\tau}d\tau$$

$$r(\tau) = \frac{1}{2\pi i}\int \phi(s)e^{s\tau}ds$$

$$\tilde{\phi}(s) = \int_0^{\infty} r(\tau)e^{-s\tau}d\tau = \mathcal{L}r(\tau)$$





## Continuous time models – properties of the spectrum

- $\phi^*(s) = \phi(-\overline{s})$
- $\phi(i\omega)$ nonnegative definite
- Diagonal elements of $\phi(i\omega)$ real–valued and positive
- Scalar case: $\phi(i\omega) = \phi(-i\omega) \Longleftrightarrow$ Real$(x(t))$ and Imag$(x(t))$ uncorrelated
- $\phi_{xy}^*(s) = \phi_{yx}(-\overline{s})$
- $\phi(s) = \tilde{\phi}(s) + \tilde{\phi}^*(-\overline{s})$





## Spectral factorization

$$\phi(i\omega) = \frac{B(i\omega)B(-i\omega)}{A(i\omega)A(-i\omega)}$$

$$A(p) = p^n + a_1p^{n-1} + \ldots + a_n$$

$$B(p) = b_1p^{n-1} + \ldots + b_n$$

all zeros in LHP.

**Linear filtering**

$$y(t) = H(p)u(t)$$

$$\phi_y(s) = H(s)\phi_u(s)H^*(-\overline{s})$$

## White noise and Wiener processes

**Definition**: $u(t)$ white noise if $\phi_u(i\omega)$ constant

$$r(0) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(i\omega) \times 1 \, d\omega = \infty$$

Try

$$r(\tau) = r_o \delta(\tau)$$

Wiener process

$$w(t) = \int_0^t e(s)ds$$

$$\begin{aligned} r(t+\tau, t) &= Ew(t+\tau)w^*(t) \\ &= r_o t \end{aligned}$$

$$Edwdw^* = r_o dt$$

Informally,

$$\frac{dw(t)}{dt} = e(t)$$





## State space models

$$\dot{x} = Ax + Be$$

$$dx = Axdt + Bdw$$

Let $\tau > 0$. Then

$$\begin{aligned} R_x(t+\tau, t) &= Ex(t+\tau)x^*(t) \\ &= e^{A\tau}P(t) \\ \dot{P}(t) &= AP(t) + P(t)A^* + BR_oB^* \end{aligned}$$

For asymptotically stable systems and $t_0 \to \infty$, $P(t) \to P$

$$0 = AP + PA^* + BR_oB^*$$

(Lyapunov equation)





## Sampling stochastic models

$$\begin{aligned} \dot{x} &= Ax + v \\ r_v(\tau) &= R_c\delta(\tau) \end{aligned}$$

$$\begin{aligned} x(k+1) &= Fx(k) + e(k) \\ \text{cov}(e(k)) &= R_d \end{aligned}$$

$$\begin{aligned} F &= e^{Ah} \\ R_d &= \int_0^h e^{As}R_c e^{A^*s}ds \end{aligned}$$





## Deriving a sampled model

Continuous-time model

$$\begin{aligned} \dot{x} &= Ax + v \\ r_v(\tau) &= R_c\delta(\tau) \end{aligned}$$

Formal solution

$$\begin{aligned} x(kh+h) &= e^{Ah}x(kh) \\ &\quad + \int_{kh}^{kh+h} e^{A(kh+h-s)}v(s)ds \\ &\triangleq Fx(kh) + e(kh) \end{aligned}$$

$e(kh)$ discrete time white noise!

$$Ee(kh) = 0$$

Compute $R_d = Ee(kh)e^T(kh)$.

## Sampling and aliasing

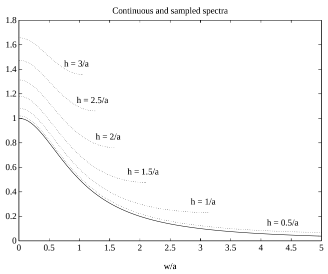
$$\phi_d(e^{i\omega h}) = \sum_{j=-\infty}^{\infty} \phi_c(i(\omega + j\frac{2\pi}{h}))$$

Special case: no signal energy above $\omega_N = \pi/h$.

$$\phi_d(e^{i\omega h}) = \phi_c(i\omega)$$

### Example

$$(p + a)y(t) = av(t)$$








## Positive real part of the spectrum

Consider the ARMA process

$$A(q)y(t) = C(q)e(t)$$

and the identity

$$\lambda^2 CC^* = DA^* + AD^*$$

**Results**

1. If $\lambda^2$ and $C$ are given, there exists a unique $D$.
2. If $D$ is given, there exists $\lambda^2$ and $C$ precisely when $D/A$ are positive real: $\left[\text{Re}D(e^{i\omega})/A(e^{i\omega}) \geq 0, \quad \text{all } \omega\right]$
3. $\tilde{\phi}(z) = D(z)/A(z)$





## Positive real part of spectrum, cont'd

ARMA process

$$A(q)y(t) = C(q)e(t)$$

Identity

$$\lambda^2 C(z)C(z^{-1}) \equiv D(z)A(z^{-1}) + A(z)D(z^{-1})$$

Spectrum

$$\begin{aligned} \lambda^2 \frac{C(z)C(z^{-1})}{A(z)A(z^{-1})} &\equiv \frac{D(z)}{A(z)} + \frac{D(z^{-1})}{A(z^{-1})} \\ \phi(z) &= \tilde{\phi}(z) + \tilde{\phi}(z^{-1}) \end{aligned}$$

Interpretation: partial fraction decomposition!





## Positive real part of spectrum, cont'd

Continuous time models

$$\begin{aligned} y(t) &= \frac{B(p)}{F(p)} e(t) \\ B(p) &= b_1 p^{n-1} + \ldots + b_n \\ F(p) &= p^n + f_1 p^{n-1} + \ldots + f_n \end{aligned}$$

$$\tilde{\phi}(s) = \frac{D(s)}{F(s)}$$

$$B(s)B(-s) = D(s)F(-s) + D(-s)F(s)$$

## Positive real part of the spectrum – state space form

State space model

$$\begin{aligned} x(t+1) &= Fx(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

Positive real part of the spectrum

$$\tilde{\phi}(z) = J + H(zI - F)^{-1}FPH^*$$

where

$$\begin{aligned} J &= (HPH^* + R_2)/2 \\ P &= FPF^* + R_1 \end{aligned}$$





## Positive real part of the spectrum – Kalman Yakobovich lemma

Define

$$\tilde{\phi}(z) = J + H(zI - F)^{-1}FPH^*$$

**Lemma**. $\tilde{\phi}(z)$ is positive real if and only if there exist $P$ (positive definite Hermitian), $L$ and $W$, such that

$$\begin{aligned} P &= FPF^* + LL^* \\ F^*PG &= H^* - LW \\ W^*W &= J + J^* - G^*PG \end{aligned}$$





## Effect of linear filtering on the bispectrum

**Zero mean Gaussian processes**

$$R(m,n) = 0, \quad B(z_1, z_2) = 0$$

**White noise**

Let $Ex(t) = 0, \quad Ex^2(t) = \sigma^2, \quad Ex^3(t) = \beta.$

$$R(m,n) = \beta\delta(m,n), \quad B(z_1, z_2) = \beta$$





## Effect of linear filtering on the bispectrum

**Filtering**

$$y(t) = \sum_{k=0}^{\infty} h_k u(t-k)$$

Input $u(t)$ zero mean.

Result:

$$B_y(z_1, z_2) = H(z_1^{-1}z_2^{-1})H(z_1)H(z_2)B_u(z_1, z_2)$$

## Algorithm for ARMA covariance computation

Process

$$
\begin{aligned}
A(q)y(t) &= C(q)e(t) \\
Ee^2(t) &= \lambda^2
\end{aligned}
$$

Compute $r(k) = Ey(t+k)y(t)$

Solve

$$\lambda^2 C(z)C(z^{-1}) \equiv D(z)A(z^{-1}) + A(z)D(z^{-1})$$

Recall

$$\tilde{\phi}(z) = \frac{D(z)}{A(z)} = \sum_{k=1}^{\infty} r(k)z^{-k} + \frac{1}{2}r(0)$$





## Algorithm for ARMA covariance computation, cont'd

Algorithm

$$
\begin{aligned}
r(0) &= 2d_o \\
r(1) &= d_1 - d_o a_1 \\
r(2) &= d_2 - d_o a_2 - a_1 r(1) \\
&\vdots \\
r(n) &= d_n - d_o a_n - a_1 r(n-1) - a_2 r(n-2) \\
&\quad \ldots - a_{n-1} r(1) \\
r(k) &= -\sum_{i=1}^{n} a_i r(k-i) \quad k > n
\end{aligned}
$$

(Yule Walker equations)





## Algorithm – continuous time covariance function

Given model

$$
\begin{aligned}
y(t) &= \frac{B(p)}{F(p)}e(t) \\
B(p) &= b_1 p^{n-1} + \ldots + b_n \\
F(p) &= p^n + f_1 p^{p-1} + \ldots + f_n
\end{aligned}
$$

1. Compute $D(s)$ from

$$B(s)B(-s) = D(s)F(-s) + D(-s)F(s)$$





## Algorithm – continuous time covariance function

Recall

$$\tilde{\phi}(s) = \mathcal{L}r(t) = \frac{D(s)}{F(s)}$$

2. Choose $H$, $F_c$ and $x_o$ so that

$$H(sI - F_c)^{-1}x_o = \frac{D(s)}{F(s)}$$

$$
\begin{aligned}
\dot{x} &= F_c x \quad x(0) = x_o \\
r &= Hx
\end{aligned}
$$

$$r(t) = He^{F_c t}x_o \qquad (t \geq 0)$$

## Algorithm for sampling

Given model

$$A(p)y(t) = B(p)e(t)$$

where

$$\begin{aligned} A(p) &= p^n + a_1 p^{n-1} + \ldots + a_n \\ B(p) &= b_1 p^{n-1} + \ldots + b_n \\ Ee(t)e(s) &= \delta(t-s) \end{aligned}$$

*Algorithm*

**Step 1**. Compute $\tilde{\phi}_c(s) = D(s)/A(s)$

$$D(s) = d_1 s^{n-1} + \ldots + d_n$$

from

$$B(s)B(-s) \equiv D(s)A(-s) + D(-s)A(s)$$





## Algorithm for sampling, cont'd

**Step 2**. Use

$$\begin{aligned} \tilde{\phi}_c(s) &= \mathcal{L}\{r(t)\} \\ \tilde{\phi}_d(z) &= [\mathcal{Z}\{r(kh)\} - \frac{1}{2}r(0)] \end{aligned}$$

to compute

$$\tilde{\phi}_d(z) = h\frac{D_d(z)}{A_d(z)}$$

**Step 3**. Spectral factorization

$$\lambda^2 C(z)C(z^{-1}) = D_d(z)A_d(z^{-1}) + A_d(z)D_d(z^{-1})$$





## An algorithm for solving Lyapunov equation

Equation

$$P = FPF^T + R_1$$

*Bad idea*: Iterate

$$P(t+1) = FP(t)F^T + R_1$$

until convergence (slow!)

*Acceleration algorithm*

Compute $P(2^k)$, $k = 1, 2, \ldots$

**1**. Set $P_1 = R_1 \quad F_1 = F \quad k = 1$.

**2**. Iterate until convergence

$$\begin{aligned} P_{k+1} &= F_k P_k F_k^T + P_k \\ F_{k+1} &= F_k^2 \end{aligned}$$