Stochastic dynamic systems
Chapter 3

Models

- Stochastic processes
- Markov processes
- Covariance functions and spectrum
- Bispectrum

Stochastic processes

$\{x(t)\}$, $t \in \mathbf{N}$ (a family of random variables)

Full description

$$P(x(t_1) \le x_1, x(t_2) \le x_2, \ldots, x(t_k) \le x_k)$$

Strictly stationary if

$$P(x(t_1 + \tau) \le x_1, \ldots, x(t_k + \tau) \le x_k)$$

does not depend on τ.

Stationary stochastic processes

Mean

$$m(t) = m_x(t) \triangleq Ex(t)$$

Autocovariance function

$$r(t,s) = r_x(t,s) \triangleq E[x(t) - m_x(t)][x(s) - m_x(s)]^T$$

Cross–covariance function of $x(t)$ and $y(t)$

$$r_{xy}(t,s) \triangleq E[x(t) - m_x(t)][y(s) - m_y(s)]^T$$

Weakly stationary process

$$\begin{aligned} m(t+\tau) &= m(t) \\ r(t+\tau, s+\tau) &= r(t,s) \end{aligned} \qquad \text{all } \tau$$

Convention

$$r(\tau) \triangleq E[x(t+\tau) - m][x(t) - m]^T$$

A sequence of independent, identically distributed (i.i.d) random variables is called *white noise*

ARMA models

$$y(t) + a_1 y(t-1) + \ldots + a_n y(t-n) = \\ e(t) + c_1 e(t-1) + \ldots + c_m e(t-m)$$

$$\begin{aligned} A(z) &= z^n + a_1 z^{n-1} + \ldots + a_n \\ C(z) &= z^m + c_1 z^{m-1} + \ldots + c_m \end{aligned}$$

shift operator q: $qx(t) = x(t+1)$

$$A(q)y(t-n) = C(q)e(t-m)$$

or, relabelled

$$A(q)y(t) = C(q)e(t)$$

ARMAX process (with input)

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

Markov processes

Definition ($t_1 < t_2 < \ldots < t_k < t$)

$$p(x(t)|x(t_1), \ldots x(t_k)) = p(x(t)|x(t_k))$$

Example

$$x(t+1) = g(x(t), t) + v(x(t), t)$$

- $\{v(x(t), t)\}$ white noise [$v(x(t_1), t_1)$ and $v(x(t_2), t_2)$ independent for $t_1 \neq t_2$]
- $g(x(t), t)$ does not depend on older values of $x(\cdot)$

The concept of state

$$x(t+1) = g(x(t), t) + v(x(t), t)$$

(v white noise).

Stochastic difference equation.

The conditional pdf $p(x(t+1)|x(t))$ is called the *hyperstate*. It is needed for a full description.

Covariance function

Definition

$$r(\tau) \triangleq Ex(t+\tau)x^*(t)$$

Cross–covariance function

$$r_{xy}(\tau) = Ex(t+\tau)y^*(t)$$

Properties

i) $r(\tau)$ generally complex–valued, $r(0)$ has real–valued and positive diagonal elements.

ii)

$$r(-\tau) = r^*(\tau)$$

Covariance function, cont'd

iii)

$$R(m) = \begin{pmatrix} r(0) & r(1) & \dots & r(m-1) \\ r(-1) & r(0) & & \\ \vdots & & \ddots & \vdots \\ r(-m+1) & & \dots & r(0) \end{pmatrix}$$

Hermitian matrix (i.e. $R^*(m) = R(m)$)

iv) $\{r(\tau)\}$ a nonnegative definite sequence (i.e. $w^* R(m) w \geq 0$ for all w)

Spectrum

Definition

$$\phi(z) = h \sum_{n=-\infty}^{\infty} r(nh) z^{-n}$$

(often $h = 1$).

Cross–spectrum

$$\phi_{xy}(z) = \sum_{n=-\infty}^{\infty} r_{xy}(n) z^{-n}$$

Properties

i) $\phi^*(z) = \phi(z^{-*})$

ii) Spectral density matrix $\phi(e^{i\omega})$ nonnegative definite.

Spectrum properties, cont'd

iii) Diagonal elements of $\phi(e^{i\omega})$ are real–valued and positive.

iv) Scalar case: $\phi(e^{i\omega})$ in general *not* symmetric. If Real($x(t)$), Imag($x(t)$) uncorrelated, $\phi(e^{i\omega})$ symmetric [$\phi(e^{-i\omega}) \equiv \phi(e^{i\omega})$]

Corollary

$$\phi_{xy}^*(z) = \phi_{yx}(z^{-*})$$

Positive real part of spectrum

Definition

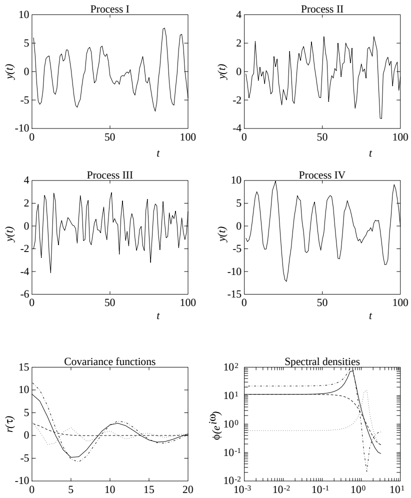
$$\tilde{\phi}(z) \triangleq \frac{1}{2} r(0) + \sum_{n=1}^{\infty} r(n) z^{-n}$$

Properties

$$\phi(z) = \tilde{\phi}(z) + \tilde{\phi}^*(z^{-*})$$

$$\phi(e^{i\omega}) = \tilde{\phi}(e^{i\omega}) + \tilde{\phi}^*(e^{i\omega})$$

Example – ARMA processes



Process I (solid): $A(q) = q^2 - 1.5q + 0.8$, $C(q) = 1$.
Process II (dashed): $A(q) = q^2 - 1.0q + 0.3$, $C(q) = 1$.
Process III (dotted): $A(q) = q^2 - 0.5q + 0.8$, $C(q) = 1$.
Process IV (dash-dotted): $A(q) = q^2 - 1.0q + 0.8$, $C(q) = q^2 - 0.5q + 0.9$.

Bispectrum

Simplest example of polyspectrum

Third moment sequence of stationary process $x(t)$:

$$R(m,n) = Ex(t)x(t+m)x(t+n)$$

Bispectrum

$$B(z_1,z_2) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} R(m,n) z_1^{-m} z_2^{-n}$$

$R(m,n)$ and $B(z_1,z_2)$ satisfy a number of symmetry relations. They are uniquely given from a part only of the (m,n) -space [or (z_1,z_2) -space].