Stochastic dynamic systems

Chapter 2

Some probability theory

- Basic facts
- Conditional distributions
- Complex-valued Gaussian variables

Random variables and distributions

Distribution function

$$F_\xi(x) = P(\xi \leq x)$$

Properties

- $F(x)$ increasing
- $\lim_{x\to\infty} F(x) = 1$
- $\lim_{x\to-\infty} F(x) = 0$

Random variables and distributions, cont'd

Probability density function (pdf)

$$p_\xi(x) = \frac{dF_\xi(x)}{dx}$$

Properties

- $p_\xi(x) \geq 0$
- $\int_{-\infty}^{\infty} p_\xi(x)dx = 1$
- $P(a < \xi \leq b) = \int_a^b p_\xi(x)dx$

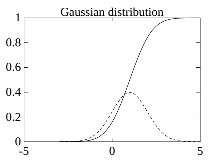
Examples

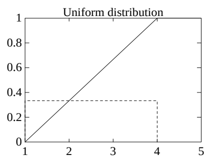
Gaussian distribution

$$p_\xi(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-m)^2}{2\sigma^2}} \qquad (\sigma > 0, \quad x \in \mathbf{R})$$

Uniform distribution

$$p_\xi(x) = \begin{cases} \frac{1}{b-a} & a < x \le b \quad (b > a) \\ 0 & \text{elsewhere} \end{cases}$$





Multivariate stochastic variables

Joint distribution function

$$F_{\xi,\eta}(x, y) = P(\xi \le x, \eta \le y)$$

Probability density function

$$p_{\xi,\eta}(x, y) = \frac{\partial^2 F_{\xi,\eta}(x, y)}{\partial x \partial y}$$

Properties

- $F_{\xi,\eta}(x, y)$ is increasing in x and in y
- $F_{\xi,\eta}(x, y) \to 1$ as $x \to \infty$ and $y \to \infty$
- $F_{\xi,\eta}(x, y) \to 0$ as $x \to -\infty$ and $y \to -\infty$
- $p((\xi, \eta) \in A) = \int\int_A p_{\xi,\eta}(x, y) dx dy$

Multivariate stochastic variables, cont'd

Properties

- $F_{\xi,\eta}(x, y) = \int_{-\infty}^{y} \int_{-\infty}^{x} p_{\xi,\eta}(x', y') dx' dy'$
- $F_\xi(x) = \lim_{y \to \infty} F_{\xi,\eta}(x, y)$
- $p_\xi(x) = \int_{-\infty}^{\infty} p_{\xi,\eta}(x, y) dy$

Moments of random variables

Mean value

$$m = E\xi = \int x p_\xi(x) dx$$

Covariance matrix

$$\text{cov}(\xi) = E(\xi - m)(\xi - m)^T = E\xi\xi^T - mm^T$$

More generally

$$Eg(\xi) = \int g(\xi) p_\xi(x) dx$$

Result

$Ex = m$, $\text{cov}(x) = P \Rightarrow$
$Ex^T Sx = m^T Sm + \text{tr} SP$

Gaussian distributions

$$\xi \sim N(m, P)$$

$$p_\xi(x) \triangleq \frac{1}{(2\pi)^{n/2}(\det P)^{1/2}} e^{-\frac{1}{2}(x-m)^T P^{-1}(x-m)}$$

Properties

- If $\xi \sim N(m, P)$, then $E\xi = m$
- If $\xi \sim N(m, P)$, then $\text{cov}(\xi) = P$
- If $\xi \sim N(m, P)$, A an $r|n$ matrix of rank $r \le n$, then $\eta = A\xi + b \sim N(Am + b, APA^T)$

Correlation and dependence

ξ and η two random vectors

- are uncorrelated if
$$E(\xi - m_\xi)(\eta - m_\eta)^T = 0$$
- are independent if
$$p_{\xi,\eta}(x, y) = p_\xi(x) p_\eta(y)$$

Results

- ξ, η independent $\Longrightarrow$ ξ, η uncorrelated
- ξ, η uncorrelated and Gaussian $\Longrightarrow$ ξ, η independent

Conditional distributions

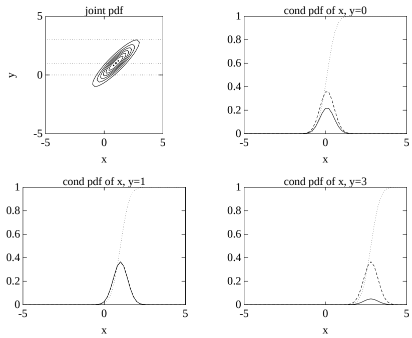
Bayes' rule

$$p_{\xi|\eta=y}(x) = \frac{p_{\xi,\eta}(x, y)}{p_y(y)}$$

Result

$$E\xi = E_\eta\{E[\xi|\eta]\}$$

Example



Conditional means for Gaussian variables

Result 1

$$\begin{pmatrix} x \\ y \end{pmatrix} \sim \mathbf{N}\left(\begin{pmatrix} m_x \\ m_y \end{pmatrix}, \begin{pmatrix} R_x & R_{xy} \\ R_{yx} & R_y \end{pmatrix} \right)$$
$$\Rightarrow$$
$$p(x|y) \sim \mathbf{N}(m, P)$$

with

$$m = \hat{x} = E[x|y] = m_x + R_{xy} R_y^{-1}(y - m_y)$$

$$P = \text{cov}[x|y] = R_x - R_{xy} R_y^{-1} R_{yx}$$

Conditional means for Gaussian variables, cont'd

Result 2

$x - \widehat{x}$ and y are uncorrelated

Result 3

$\begin{pmatrix} x \\ y \end{pmatrix}$ jointly Gaussian distributed, $y = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$, y_1 and y_2 uncorrelated. Then

$$E[x|y] = E[x|y_1] + E[x|y_2] - E[x]$$

Conditional means for Gaussian varables — Proofs of the results

Result 2: Direct from Result 1.

Result 3: Easy from Result 1 (use R_y diagonal)

Result 1: Use defintion of conditional pdf, conditional mean, and results for partitionated matrices.