Stochastic dynamic systems
Chapter 11

Linear quadratic Gaussian (LQG) control

- The optimal controllers
- Duality
- Closed loop properties
- Controller design

LQG problem formulation

System

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) + v(t) \\ y(t) &= Hx(t) + e(t) \end{aligned}$$

Disturbances Gaussian, zero mean and

$$E \begin{pmatrix} v(t) \\ e(t) \end{pmatrix} (v^T(t) \quad e^T(t)) = \begin{pmatrix} R_1 & R_{12} \\ R_{21} & R_2 \end{pmatrix}$$

Criterion (performance index)

$$\begin{aligned} \ell \; = \; & x^T(N) Q_o x(N) \\ & + \textstyle\sum_{t=t_o}^{N-1} (x^T(t) \quad u^T(t)) \begin{pmatrix} Q_1 & Q_{12} \\ Q_{21} & Q_2 \end{pmatrix} \begin{pmatrix} x(t) \\ u(t) \end{pmatrix} \end{aligned}$$

LQG problem formulation, cont'd

Find the controller that minimizes $E\ell$.

Three cases:

1. Deterministic systems
2. Complete state information
   ($H = I, \quad e(t) \equiv 0$)
3. Incomplete state information

LQG lemma

Riccati equation (assume $S(t) \geq 0$)

$$\begin{aligned} S(t) &= F^T S(t+1)F + Q_1 - [F^T S(t+1)G + Q_{12}] \\ &\quad \times [G^T S(t+1)G + Q_2]^{-1}[G^T S(t+1)F + Q_{21}] \\ S(N) &= Q_o \end{aligned}$$

Gain matrix

$$L(t) = [G^T S(t+1)G + Q_2]^{-1}[G^T S(t+1)F + Q_{21}]$$

Result

$$\begin{aligned} \ell &= x^T(t_o)S(t_o)x(t_o) \\ &\quad + \sum_{t=t_1}^{N-1} [u(t) + L(t)x(t)]^T \\ &\quad \times [G^T S(t+1)G + Q_2][u(t) + L(t)x(t)] \\ &\quad + \sum_{t=t_1}^{N-1} \Big(2v^T(t)S(t+1)[Fx(t) + Gu(t)] \\ &\qquad + v^T(t)S(t+1)v(t)\Big) \end{aligned}$$

LQG: Deterministic case

Assume no process noise ($v(t) \equiv 0$) and complete state information ($y(t) \equiv x(t)$). Then the optimal controller is

$$u(t) = -L(t)x(t)$$

The minimal loss is

$$\min V = m^T S(t_o)m + \text{tr}S(t_o)R_o$$

LQG: Complete state information

The optimal controller is

$$u(t) = -L(t)x(t)$$

The minimal loss is

$$\min V = m^T S(t_o)m + \text{tr}S(t_o)R_o + \sum_{t=0}^{N-1} \text{tr}S(t+1)R_1$$

LQG: Incomplete state information

The optimal controller is

$$u(t) = -L(t)\hat{x}(t|t-k)$$

The minimal loss is

$$\begin{aligned} \min E\ell &= m^T S(t_o)m + \text{tr } S(t_o)R_o \\ &\quad + \textstyle\sum_{t=t_o}^{N-1} \text{tr } S(t+1)R_1 \\ &\quad + \textstyle\sum_{t=t_o}^{N-1} \text{tr } \{P(t|t-k)L^T(t) \\ &\qquad \times [G^T S(t+1)G + Q_2]L(t)\} \end{aligned}$$

LQG: Duality between estimation and control

An optimal state estimation problem can be turned into an optimal control problem:

$$
\begin{array}{rcl}
\textit{Control} & & \textit{Estimation} \\
\\
\bar{t} & = & t_o + N - t \\
\overline{F} & = & F^T \\
\overline{G} & = & H^T \\
\overline{Q}_o & = & R_o \\
\overline{Q}_1 & = & R_1 \\
\overline{Q}_{12} & = & R_{12} \\
\overline{Q}_2 & = & R_2
\end{array}
$$

LQG: Closed loop properties

Consider the asymptotic case
$(t_o \to -\infty,\ N \to \infty)$

System

$$
\begin{aligned}
x(t+1) &= Fx(t) + Gu(t) + v(t) \\
y(t) &= Hx(t) + e(t)
\end{aligned}
$$

Controller

$$
\begin{aligned}
u(t) &= -L\hat{x}(t|t-1) + u_r(t) \\
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) \\
&\quad + K_p[y(t) - H\hat{x}(t|t-1)]
\end{aligned}
$$

Closed loop system

$$
\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1|t) \end{pmatrix} = \begin{pmatrix} F - GL & GL \\ 0 & F - K_pH \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t|t-1) \end{pmatrix}
+ \begin{pmatrix} G \\ 0 \end{pmatrix} u_r(t) + \begin{pmatrix} I & 0 \\ I & -K_p \end{pmatrix} \begin{pmatrix} v(t) \\ e(t) \end{pmatrix}
$$

LQG: Closed loop properties

Consider the asymptotic case
$(t_o \to -\infty,\ N \to \infty)$

System

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\begin{aligned}
x(t+1) &= Fx(t) + Gu(t) + v(t) \\
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\end{aligned}
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Controller

$$
\begin{aligned}
u(t) &= -L\hat{x}(t|t) + u_r(t) \\
\hat{x}(t+1|t) &= F\hat{x}(t|t-1) + Gu(t) \\
&\quad + K_p[y(t) - H\hat{x}(t|t-1)] \\
\hat{x}(t|t) &= \hat{x}(t|t-1) + K_f[y(t) - H\hat{x}(t|t-1)].
\end{aligned}
$$

Closed loop system

$$
\begin{pmatrix} x(t+1) \\ \tilde{x}(t+1|t) \end{pmatrix} = \begin{pmatrix} F - GL & GL(I - K_fH) \\ 0 & F - K_pH \end{pmatrix} \begin{pmatrix} x(t) \\ \tilde{x}(t|t-1) \end{pmatrix}
+ \begin{pmatrix} G \\ 0 \end{pmatrix} u_r(t) + \begin{pmatrix} I & -GLK_f \\ I & -K_p \end{pmatrix} \begin{pmatrix} v(t) \\ e(t) \end{pmatrix}
$$

LQG: Closed loop poles

Open loop characteristic polynomial

$$\phi_{ol}(z) \triangleq \det[zI - F]$$

Closed loop characteristic polynomial

$$\phi_{cl}(z) \triangleq \det[zI - (F - GL)]$$

Relation

$$\phi_{cl}(z) = \phi_{ol}(z) \det[I + L(zI - F)^{-1}G]$$

LQG: Closed loop poles

Let

$$\begin{aligned} S &= F^T S F + Q_1 - (F^T S G + Q_{12}) \\ &\quad \times (G^T S G + Q_2)^{-1}(Q_{21} + G^T S F) \\ L &= (G^T S G + Q_2)^{-1}(Q_{21} + G^T S F) \end{aligned}$$

Then

$$\begin{aligned} &[I + L(zI - F)^{-1}G]^T [G^T S G + Q_2] \\ &\quad \times [I + L(z^{-1}I - F)^{-1}G] \end{aligned}$$

$$\begin{aligned} &\equiv Q_2 + G^T(zI - F^T)^{-1}Q_1(z^{-1}I - F)^{-1}G \\ &\quad + G^T(zI - F^T)^{-1}Q_{12} + Q_{21}(z^{-1}I - F)^{-1}G. \end{aligned}$$

Interpretation: Spectral factorization!

Consequence:

$$\begin{aligned} &\det[Q_2 + G^T(zI - F^T)^{-1}Q_1(z^{-1}I - F)^{-1}G \\ &\quad + G^T(zI - F^T)^{-1}Q_{12} + Q_{21}(z^{-1}I - F)^{-1}G] \\ &= \det[G^T S G + Q_2] \ \frac{\phi_{c\ell}(z)\phi_{c\ell}(z^{-1})}{\phi_{o\ell}(z)\phi_{o\ell}(z^{-1})} \end{aligned}$$

LQG design

- Weighting matrices Q_o, Q_1, Q_{12}, Q_2 are seldom given, but are as used design variables.
- The controller is invariant to a scaling of all weighting matrices.
- LQG can be viewed as a particular pole placement design.
- A rule of thumb: Select $Q_{12} = 0$, Q_1 and Q_2 diagonal with

$$\begin{aligned} Q_{1,ii}^{-1} &= \max E x_i^2(t) \\ Q_{2,ii}^{-1} &= \max E u_i^2(t) \end{aligned}$$

  Adjust weights after simulating the closed loop system.

Example of LQG design

Sampled double integrator

$$\begin{aligned} x(t+1) &= \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix} x(t) + \begin{pmatrix} h^2/2 \\ h \end{pmatrix} u(t) + v(t) \\ y(t) &= (1 \ \ 0)x(t) + e(t) \end{aligned}$$

Numerical values: $h = 0.5$, $r_v = 10^{-3}$, $r_e = 10^{-3}$.

Regulator

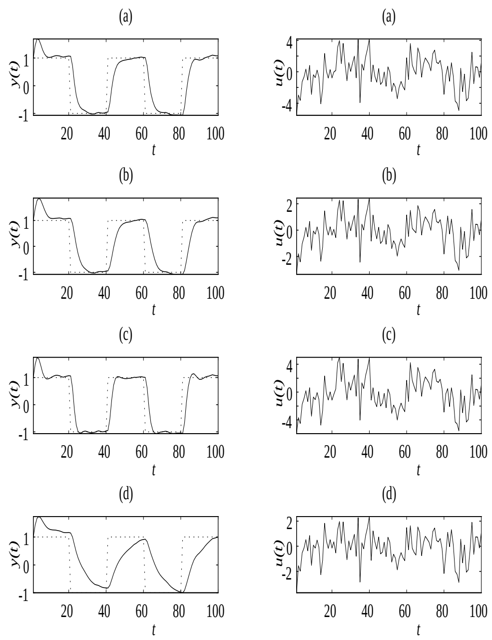
$$u(t) = -L\hat{x}(t|t) + Mr(t)$$

Example, cont'd

Cases:

1. $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q_2 = 0.1$
2. $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $Q_2 = 1$
3. $Q_1 = \begin{pmatrix} 10 & 0 \\ 0 & 1 \end{pmatrix}$, $Q_2 = 1$
4. $Q_1 = \begin{pmatrix} 1 & 0 \\ 0 & 10 \end{pmatrix}$, $Q_2 = 1$

Example, cont'd



Minimum variance control

System

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

is assumed to be minimum phase ($B(z)$ all zeros inside the unit circle).

$$\begin{aligned} A(q) &= q^n + a_1 q^{n-1} + \ldots + a_n \\ B(q) &= b_0 q^m + a_1 q^{m-1} + \ldots + b_m, \ b_0 \neq 0 \\ C(q) &= q^n + c_1 q^{n-1} + \ldots + c_n \end{aligned}$$

Pole excess $k \triangleq n - m > 0$.

Criteria

$$\begin{aligned} V_1 &= Ey^2(t+k) \\ V_2 &= E\frac{1}{M}\sum_{s=0}^{M-1} Ey^2(t+k+s) \end{aligned}$$

Minimum variance control, cont'd

Optimal controller is related to k step optimal prediction!

$$\begin{aligned} V_1 &= E[\hat{y}(t+k|t) + \tilde{y}(t+k|t)]^2 \\ &\geq E[\tilde{y}(t+k|t)]^2 \end{aligned}$$

Optimal control when $\hat{y}(t+k|t) = 0$.

Minimum variance control, cont'd

System

$$A(q)y(t) = B(q)u(t) + C(q)e(t)$$

Predictor identity

$$z^{k-1}C(z) \equiv A(z)F(z) + L(z)$$

Optimal regulator

$$u(t) = -\frac{L(q)}{B(q)F(q)}y(t)$$

Optimal performance

$$\begin{aligned} \min V_1 &= E\tilde{y}^2(t+k|t) \\ &= \lambda^2(1 + f_1^2 + \ldots + f_{k-1}^2) \end{aligned}$$

Minimum variance control, cont'd

Results

1. Minimal variance controller optimize *both* V_1 and V_2.

2. Closed loop system:

   $B(q)C(q)y(t) = B(q)C(q)F(q)e(t-k+1)$

   Note

   - $B(z)$ min phase is essential
   - $C(z)$ min phase is a mild assumption

Minimum variance control, cont'd

3 Extension to non-minimum phase systems - minimization of V_1 only. Optimal controller

$$u(t) = -\frac{P(q)}{R(q)B_+(q)}y(t)$$

where

$$B(z) = B_+(z)B_-(z)$$

$B_+(z)$ all zeros outside the unit cirle.
$r =$ deg B_-.

and

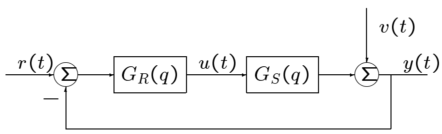
$A(z)R(z)+B_+(z)P(z) \equiv C(z)B_-(z^{-1})z^{k+r-1}$

(deg $R = k + r - 1$, deg $P = n - 1$)

Choice of observer poles

- True covariances R_1, R_2 may be unknown. Treat as design parameters instead.
- Pole placement - make observer dynamics somewhat faster than controller poles.
- Loop transfer recovery (LTR) (mimic feedback from full states by introducing high gains - use with care).

Control design considerations



Loop gain

$$G_o(q) = G_S(q)G_R(q)$$

Closed loop system

$y(t) = [I+G_o(q)]^{-1}v(t)+[I+G_o(q)]^{-1}G_o(q)r(t)$

Control design considerations, cont'd

- *Servo properties*: $y(t)$ should follow $r(t)$. Hence
$$[I + G_o(e^{i\omega})]^{-1} \quad \text{small}$$
for the dominating frequency band of $r(t)$.

- *Damping of disturbances*.
$$[I + G_o(e^{i\omega})]^{-1} \quad \text{small}$$
for the dominating frequency band of $v(t)$.

- *Stabililty robustness*. The Nyquist curve should never be close to -1.

- *Sensitivity to model uncertainties*.
$$[I + G_o(e^{i\omega})]^{-1} \quad \text{small}$$
in the frequency band of uncertain dynamics.

Loop transfer recovery (LTR)

Deterministic system

$$\begin{aligned} x(t+1) &= Fx(t) + Gu(t) \\ y(t) &= Hx(t) \end{aligned}$$

Assume

- dim u = dim y, and HG is invertible.

- The open loop transfer function
$$G_S(q) = H(qI - F)^{-1}G$$
has an asymptotically stable inverse.

Regulator

$$u(t) = -L\hat{x}(t|t)$$

Loop transfer recovery (LTR), cont'd

Design with LQG

Choice of covariance matrices:

$$R_1 = GG^T, R_2 = 0$$

Results

- Filter gain $K = G(HG)^{-1}$.

- Loop gain
$$G_R(q)G_S(g) = L(qI - F)^{-1}G$$
(loop transfer recovery)

Design example LTR

System (sampled harmonic oscillator)

$$\begin{aligned} x(t+1) &= \begin{pmatrix} \cos(\omega_o h) & (\sin(\omega_o h))/h \\ -\omega_o \sin(\omega_o h) & \cos(\omega_o h) \end{pmatrix} x(t) \\ &\quad + \begin{pmatrix} (1 - \cos(\omega_o h))/\omega_o^2 \\ (\sin(\omega_o h))/\omega_o \end{pmatrix} u(t) \\ y(t) &= (\omega_o^2 \quad 0) x(t) \end{aligned}$$

Numerical parameters: $\omega_0 = 0.2,\ h = 0.5$.

Design of feedback gain

$$Q_1 = I,\ Q_2 = 1$$

Design example LTR, cont'd

Compare the cases:

a) Full state feedback

b) LTR

c) Modified LTR,

$$R_1 = GG^T, \quad R_2 = 10^{-3}$$

Design example LTR, cont'd

Closed loop response