Stochastic dynamic systems

Chapter 10

Optimal stochastic control

- Some simple examples
- Mathematical preliminaries
- Dynamic programming
- Some stochastic controllers

Optimal stochastic control - some illustrative examples

System

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) \\ y(t) &= x(t) \end{aligned}$$

Criterion

$$J = Ex^2(N)$$

Introduce stochastic aspects. Some possibilities:

- process noise
- measurement noise
- a and/or b random parameter

Optimal stochastic control - Deterministic system

Criterion

$$J = x^2(N) = [ax(N-1) + bu(N-1)]^2$$

J is minimized (and $J = 0$) if

$$u(N-1) = -\tfrac{a}{b}x(N-1)$$

$$u(0), \ldots, u(N-2) \text{ arbitrary}$$

Simple solution (dead-beat)

$$u(t) = -\frac{a}{b}x(t)$$

Optimal stochastic control - Random time constant

Assume $a(t)$ is a sequence of i. i. d. random numbers, with known mean m and variance σ^2.

Criterion

$$J = Ex^2(N) = E[E[x^2(N)|X^{N-1}, U^{N-1}]]$$

(outer expectation with respect to X^{N-1} $= \{x(0), \ldots, x(N-1)\}$)

Inner expectation

$$\begin{aligned}&E[x^2(N)|X^{N-1}, U^{N-1}]\\&= E[a(N-1)x(N-1) + bu(N-1)]^2\\&= [mx(N-1) + bu(N-1)]^2 + \sigma^2 x^2(N-1)\end{aligned}$$

Random time constant, cont'd

Optimal control action

$$u(N-1) = -\frac{m}{b}x(N-1)$$

leads to

$$J = E\sigma^2 x^2(N-1) = \sigma^2 Ex^2(N-1)$$

Repeat the reasoning. Resulting optimal control

$$u(t) = -\frac{m}{b}x(t)$$

Optimal stochastic control - Noisy observations

System

$$\begin{aligned}x(t+1) &= ax(t) + bu(t)\\y(t) &= x(t) + e(t)\end{aligned}$$

Set

$$m(t) = E[x(t)|Y^t], \quad \sigma^2(t) = \text{var}[x(t)|Y^t]$$

Criterion

$$J = Ex^2(N) = E[E[x^2(N)|Y^{N-1}]]$$

(outer expectation w. r. t. Y^{N-1})

Noisy observations, cont'd

Loss

$$\begin{aligned}&E[x^2(N)|Y^{N-1}]\\&= E[\{ax(N-1) + bu(N-1)\}^2|Y^{N-1}]\\&= [am(N-1) + bu(N-1)]^2 + a^2\sigma^2(N-1)\end{aligned}$$

Optimal controller

$$u(N-1) = -\frac{a}{b}m(N-1)$$

combined with a Kalman filter for updating $m(t)$ and $\sigma^2(t)$.

Optimal stochastic control - Process noise

System

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) + v(t) \\ y(t) &= x(t) \end{aligned}$$

$$J = E[E[x^2(N)|X^{N-1}]]$$

Inner expectation

$$\begin{aligned} &E[x^2(N)|X^{N-1}] \\ &= E[\{ax(N-1) + bu(N-1) + v(N-1)\}^2|X^{N-1}] \\ &= [ax(N-1) + bu(N-1)]^2 + r_1 \end{aligned}$$

Optimal control

$$u(N-1) = -\frac{a}{b}x(N-1)$$

Optimal stochastic control - Unknown time constant and measurement noise

System

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) \\ y(t) &= x(t) + e(t) \end{aligned}$$

a random variable, $Ea = m$, $E(a-m)^2 = \sigma^2$.

Conjecture: Optimal control is

$$\begin{aligned} u(N-1) &= -\frac{\widehat{m}(N-1)}{b}\widehat{x}(N-1) \\ \widehat{m}(N-1) &= E[a|Y^{N-1}] \\ \widehat{x}(N-1) &= E[x(N-1)|Y^{N-1}] \end{aligned}$$

Unknown time constant and measurement noise, cont'd

The conjecture is *false*. The optimal controller is

$$u(N-1) = -\frac{1}{b}\widehat{ax}(N-1)$$

Note that

$$\begin{aligned} \widehat{ax}(N-1) &= E[ax(N-1)|Y^{N-1}] \\ &\neq E[a|Y^{N-1}]E[x|Y^{N-1}] \\ &= \widehat{a}(N-1)\widehat{x}(N-1) \end{aligned}$$

Optimal stochastic control - Unknown gain

System

$$\begin{aligned} x(t+1) &= ax(t) + bu(t) + v(t) \\ y(t) &= x(t) \end{aligned}$$

b random variable, nonzero mean, finite variance.

The certainty equivalence regulator

$$\begin{aligned} u(t) &= -\frac{a}{\beta(t)}x(t) \\ \beta(t) &= E[b|Y^t] \end{aligned}$$

gives

$$E[x^2(N)|X^{N-1}] = p(N-1)\left(\frac{ax(N-1)}{\beta(N-1)}\right)^2 + r_1$$

Unknown gain, cont'd

The **cautious** regulator

$$u(t) = -\frac{a\beta(t)}{\beta^2(t) + p(t)} x(t)$$

gives

$$\begin{aligned} &E[x^2(N)|X^{N-1}] \\ &= p(N-1)\frac{(ax(N-1))^2}{\beta^2(N-1)+p(N-1)} + r_1 \end{aligned}$$

which is smaller.

Mathematical preliminaries

Lemma

Let $\ell(x, y, u)$ have a unique minimum

$$u = u^o(x, y)$$

for all x and y. Then

$$\begin{aligned} \min_{u(x,y)} E\ell(x, y, u) &= E\ell(x, y, u^o(x, y)) \\ &= E \min_u \ell(x, y, u) \end{aligned}$$

Mathematical preliminaries

Lemma

Let $f(y, u) \triangleq E[\ell(x, y, u)|y]$ have a unique minimum,

$$u = u^o(y)$$

for all y. Then

$$\min_{u(y)} E\ell(x, y, u) = E_y\{\min_u E[\ell(x, y, u)|y]\}$$

Dynamic programming – deterministic case

System

$$x(t+1) = f(x(t), u(t))$$

Criterion

$$J = \sum_{t=0}^{N-1} \ell(t, x(t), u(t))$$

Optimize J with respect to functions $u(t) = u(x(t))$.

Hamilton-Jacobi-Bellman equation (Principle of optimality)

$$V(x(t), t) = \min_{u(t)} [\ell(x(t), u(t)) + V(f(x(t), u(t)), t+1)]$$

$V(x(t), t)$ is the minimal loss, when starting at time t in the state $x(t)$.

Dynamic programming – stochastic case I

System

$$x(t+1) = f(x(t), u(t)) + v(t)$$

Criterion

$$J = E \sum_{t=0}^{N-1} \ell(t, x(t), u(t))$$

Optimize J with respect to functions $u(t) = u(x(t))$ (complete state information).

Hamilton-Jacobi-Bellman equation (Principle of optimality)

$$V(x(t),t) = \min_{u(t)} E\left[\ell(x(t),u(t)) + V(f(x(t),u(t)), t+1)|x(t)\right]$$

$V(x(t), t)$ is the expected minimal loss when starting at time t in the state $x(t)$.

Some controllers for stochastic systems

1. *Dual control* (input should both minimize the criterion and increase information about unknown system parameters).

2. *Certainty equivalence control* (replace unknown quantities by their estimates)

3. *Cautious control* (the regulator will take the uncertainty of the process parameters into account)

Dynamic programming – stochastic case II

System

$$\begin{aligned} x(t+1) &= f(x(t), u(t)) + v(t) \\ y(t) &= h(x(t)) + e(t) \end{aligned}$$

Criterion

$$J = E \sum_{t=0}^{N-1} \ell(t, x(t), u(t))$$

Optimize J with respect to functions $u(t) = u(y(t))$ (incomplete state information).

Hamilton-Jacobi-Bellman equation

$$V(\xi(t), t) =$$
$$\min_{u(t)} E\left[\ell(x(t), u(t)) + V(\xi(t+1), t+1)|Y^{t-1}\right]$$

where $\xi(t)$ is the hyperstate, describing $p(x(t)|Y^{t-1})$.